INITIAL VALUE PROBLEM FOR THE CONSTANT MEAN CURVATURE EQUATION IN THE REISSNER–NORDSTRÖM SPACETIME

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We characterize all spacelike and spherically symmetric constant mean curvature hypersurfaces in the maximally extended Reissner–Nordström spacetimes. These characterizations also provide a proof of the existence and uniqueness of the initial value problem for the spacelike and spherically symmetric constant mean curvature equation in the maximally extended Reissner–Nordström spacetimes.

Keywords: Reissner–Nordström spacetime, constant mean curvature hypersurfaces.

1. Introduction

Spacelike constant mean curvature (CMC) hypersurfaces in spacetime are interesting and important geometric objects in general relativity. Brill, Cavallo, and Isenberg proved that a spacelike CMC hypersurface in spacetime has extremal surface area among fixed enclosed volume [1]. In addition, CMC hypersurfaces are widely used in the analysis of Einstein constraint equations [3, 6]. In cosmology, CMC foliation is identified as the absolute time function [16]. Some CMC foliation theory in cosmological spacetimes can be found in [2, 14]. There are also interesting results on CMC foliations in spatially noncompact spacetime such as the Schwarzschild spacetime [7, 8, 12, 13].

In this article we consider spacelike, spherically symmetric, constant-mean curvature hypersurfaces (we use the abbreviation SSCMC hypersurfaces in convenience) in the maximally extended Reissner–Nordström spacetime. The Reissner–Nordström spacetime is the simplest nontrivial static solution of the Einstein–Maxwell field equations. For some physical reasons, cosmic censorship to exclude naked singularities for example, the Reissner–Nordström spacetime with charge smaller than mass is much more interesting than other cases, so we will focus on SSCMC hypersurfaces in this spacetime model.

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This paper will answer the existence and uniqueness of the initial value problem for the SSCMC equation in the maximally extended Reissner–Nordström spacetime. Here we state two main results in the following comprehensive way.

CHARACTERIZATION THEOREM. *All spacelike and spherically symmetric constant mean curvature hypersurfaces in the maximally extended Reissner–Nordström spacetime with the charge smaller than the mass can be characterized by two parameters c and $\bar{c}$, which are two constants of integration of the solution for the constant mean curvature equation in the standard Reissner–Nordström coordinates.*

MAIN THEOREM. *The initial value problem for the spacelike, spherically symmetric, constant mean curvature equation in the maximally extended Reissner–Nordström spacetime with the charge smaller than the mass is solvable and the solution is unique.*

The precise mathematical settings of the characterization theorem and the initial value problem (main theorem) are discussed in Section 4 and in Section 5, respectively.

Remark that Brill, Cavallo, and Isenberg in [1] gave some general discussions on SSCMC hypersurfaces in spherically symmetric static spacetime. Tuite and Ó. Murchadha also studied SSCMC hypersurfaces in the Reissner–Nordström spacetime [15]. Compared with these research projects, here we provide different analysis approaches, dress more geometric insights (figures), and show more properties of SSCMC hypersurfaces in this topic.

The motivation for studying SSCMC hypersurfaces in such a detailed way is that we hope to answer the problem of existence of CMC foliations in the Reissner–Nordström spacetime. In previous experience [7, 8, 10, 11], we have successfully constructed CMC foliations in the Schwarzschild spacetime. When facing SSCMC hypersurfaces and CMC foliation problems in the Reissner–Nordström spacetime, we have to overcome two difficulties. One difficulty is that the Penrose diagram of the maximally extended Reissner–Nordström spacetime consists of infinite many Reissner–Nordström spacetime regions. It is more complicated than the Schwarzschild spacetime (only four regions). We need a thorough discussion on glueing of two adjacent spacetimes and two adjacent SSCMC hypersurfaces.

The other difficulty is that the Reissner–Nordström spacetime metric has one more higher-order term in radius part than the Schwarzschild metric. We have to do some adjustments near the horizon to make sure our approaches also work in this new model. Fortunately, all SSCMC hypersurfaces can be characterized as well, and we expect these arguments will give a good understanding on CMC foliation problems.

The organization of this paper is as follows. We first give a brief introduction to the maximally extended Reissner–Nordström spacetime in Section 2. In Section 3, we study SSCMC hypersurfaces in each region and analyze their asymptotic behaviours, especially at infinity, coordinate singularities, and spacetime singularities. The characterization of SSCMC hypersurfaces are discussed in Section 4. We will set up the initial value problem for the SSCMC equation and prove its existence and uniqueness in Section 5.

2. Preliminaries

The Reissner–Nordström (RN) spacetime (L^4, ds^2) is a time-oriented, four-dimensional Lorentzian manifold with the metric

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \frac{1}{\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2, \qquad (1)$$

where $m > 0$ is called the mass, and e is called the charge. If $e = 0$, the spacetime reduces to the Schwarzschild spacetime.

In this paper, we mainly concentrate on the case $e^2 < m^2$. Denote the function

$$h(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} = \frac{r^2 - 2mr + e^2}{r^2}.$$

The equation $r^2 - 2mr + e^2 = 0$ has two distinct real roots, called $r_\pm = m \pm \sqrt{m^2 - e^2}$. Although $h(r_\pm) = 0$ and it looks that the metric (1) has singularities at $r = r_\pm$, after coordinates change, we know that $r = r_\pm$ are coordinates singularities. On the construction of the maximally extended RN spacetime we refer to the note [5, 9] for more detailed discussions. Here we only describe the structure of the Penrose diagram of the maximally extended RN spacetime in Fig. 1.

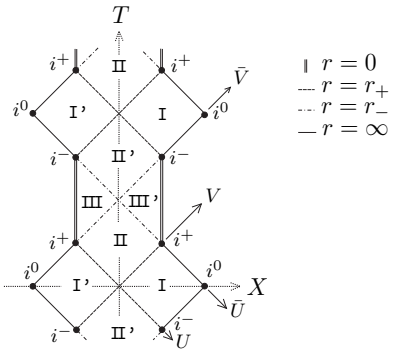


Fig. 1. Penrose diagram for the maximally extended RN spacetime with $e^2 < m^2$.

To construct this diagram, first we take one family of RN spacetimes. Each spacetime in this family is divided into three regions $r > r_+$, $r_- < r < r_+$, and $0 < r < r_-$ in the standard coordinates (t, r, θ, ϕ). There is one-to-one and onto correspondence from $r > r_+$ to region I, from $r_- < r < r_+$ to region II, and from $0 < r < r_-$ to region III, respectively. In the Penrose diagram, the spacetime metric can be smoothly extended at the interface of region I and II, or region II and III. Next, taking another family of RN spacetimes, we establish and label the correspondence from three regions $r > r_+$, $r_- < r < r_+$, $0 < r < r_-$ of each RN spacetime to regions I', II', and III', respectively. Notice that we choose opposite timelike direction in the coordinates (t, r, θ, ϕ) for the second family of

RN spacetimes. The maximally extended RN spacetime is constructed by gluing two families of RN spacetimes consecutively at r_+ for I and II', or I' and II; at r_- for II and III', or III and II', respectively.

In this article, we will set (T, X) coordinates as in Fig. 1 and then choose ∂_T as future-directed timelike vector field of the maximally extended RN spacetime. The metric of the maximally extended RN spacetime can be formally written as

$$ds^2 = P^2(T, X)(-dT^2 + dX^2) + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2,$$

where $P(T, X)$ is a smooth function and of nonlinear relations with r and t.

Let $\Sigma : (T = F(X), X, r, \theta)$ be an SSCMC hypersurface. The spacelike condition is equivalent to $(F'(X))^2 < 1$, and the mean curvature equation will be

$$F'' + (1 - (F')^2)\left(\frac{2r_T}{r} + \frac{P_T}{P} + \left(\frac{r_X}{r} + \frac{P_X}{P}\right)F'\right) - 3HP(1 - (F')^2)^{\frac{3}{2}} = 0, \qquad (2)$$

where $F' = F'(X)$, $F'' = F''(X)$, and the subscription means the partial derivative with respect to the variable. The computation of this equation is derived in the note [9] and we skip the computation here.

Our goal in this article is to analyze solutions of Eq. (2) for constant mean curvature H. However, since functions $r = r(T, X) = r(F(X), X)$ and $P = P(T, X) = P(F(X), X)$ are of complicated nonlinear relations with $T = F(X)$ and X, it is not easy to characterize solutions through Eq. (2). Instead, we will start from the SSCMC equation in the standard coordinates (t, r, θ, ϕ) of the RN spacetime to explore the properties of SSCMC hypersurfaces in the standard coordinates and then discuss the relations between the coordinates (t, r, θ, ϕ) and (T, X, θ, ϕ).

As a remark, when we analyze SSCMC hypersurfaces in Section 4, we use (U, V) and $(\bar{U}, \bar{V})$ coordinates. They both have linear relations with respect to (X, T) coordinates by $U = X - T$, $V = X + T$, and $\bar{U} = X - T + \pi/2$, $\bar{V} = X + T + \pi/2$. Fig. 1 also shows the UV and $\bar{U}\bar{V}$-axes. More discussions on coordinates changes can be found in the note [9].

3. SSCMC equation in Reissner–Nordström (t, r, θ, ϕ) coordinates

Assume that an SSCMC hypersurface Σ in the standard RN spacetime coordinates is of the form $(t = f(r), r, \theta, \phi)$. We are going to derive the SSCMC equation. The advantage of this process is that the SSCMC equation in standard coordinates and the solution have an explicit expression so that we can analyze the behaviour of SSCMC hypersurfaces at infinity, coordinate singularities, and spacetime singularities.

There are six different regions in the maximally extended RN spacetime. In the following subsections, we first deal with the SSCMC equation in region I and III, and then analyze asymptotic behaviours of SSCMC hypersurfaces at $r = \infty$, r_+, r_-, and 0. All SSCMC hypersurfaces in region I' and III' are similarly treated. The SSCMC equation in region II needs more discussions because ∂_r becomes timelike and ∂_t becomes spacelike. Although ∂_t and ∂_r change type, all SSCMC solutions can be characterized as well.

3.1. SSCMC hypersurfaces in region I and region III

PROPOSITION 3.1. *Suppose that* $\Sigma : (t = f(r), r, \theta, \phi)$ *is an SSCMC hypersurface in the RN spacetime which maps to the region* I *or* III. *The mean curvature equation is*

$$f'' + \left(\left(\frac{1}{h} - (f')^2 h\right)\left(\frac{2h}{r} + \frac{h'}{2}\right) + \frac{h'}{h}\right) f' - 3H\left(\frac{1}{h} - (f')^2 h\right)^{\frac{3}{2}} = 0,$$

where

$$h(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2}$$

and H *is the constant mean curvature. The solution is*

$$f'(r; H, c) = \frac{l(r; H, c)}{h(r)\sqrt{1 + l^2(r; H, c)}}, \qquad \text{where} \qquad l(r; H, c) = \frac{1}{\sqrt{h(r)}}\left(Hr - \frac{c}{r^2}\right),$$

for some constant of integration c. *After integration it gives*

$$f(r; H, c, \bar{c}) = \int_{r_{\text{ini}}}^{r} \frac{l(x; H, c)}{h(x)\sqrt{1 + l^2(x; H, c)}}\, dx + \bar{c},$$

where $\bar{c}$ *is another constant and* r_{ini} *is an initial number.*

Proof: Consider $F(t, r, \theta, \phi) = -t + f(r)$ and let Σ be a level set of the function F. We compute

$$\nabla F = \frac{1}{h(r)}\partial_t + f'(r)h(r)\partial_r.$$

The spacelike condition of Σ is equivalent to

$$-\frac{1}{h(r)} + (f'(r))^2 h(r) < 0 \Leftrightarrow (f'(r)h(r))^2 < 1.$$

Since we require ∂_T be the future-directed timelike vector field in the extended RN spacetime, we choose

$$e_4 = \frac{\nabla F}{\sqrt{-\langle \nabla F, \nabla F\rangle}} = \frac{\left(\frac{1}{h(r)}, h(r)f'(r), 0, 0\right)}{\sqrt{\frac{1}{h(r)} - (f'(r))^2 h(r)}}$$

as the future-directed unit timelike normal vector in region I or region III. Next, we choose an orthonormal frame on $T_p\Sigma$:

$$e_1 = \frac{(0, 0, 1, 0)}{r}, \qquad e_2 = \frac{(0, 0, 0, 1)}{r\sin\theta}, \qquad e_3 = \frac{(f'(r), 1, 0, 0)}{\sqrt{\frac{1}{h(r)} - (f'(r))^2 h(r)}}.$$

By direct computation, the second fundamental form of Σ in (L^4, ds^2) will be

$$h_{11} = h_{22} = \frac{1}{\left(1/h - (f')^2 h\right)^{\frac{1}{2}}} \frac{hf'}{r},$$

$$h_{33} = \frac{1}{\left(1/h - (f')^2 h\right)^{\frac{1}{2}}} \left(\frac{1}{1/h - (f')^2 h} \left(f'' + \frac{h' f'}{h} \right) + \frac{h' f'}{2} \right),$$

and $h_{ij} = 0$ for $i \neq j$. Hence the mean curvature equation becomes

$$f'' + \left(\left(\frac{1}{h} - (f')^2 h \right) \left(\frac{2h}{r} + \frac{h'}{2} \right) + \frac{h'}{h} \right) f' - 3H \left(\frac{1}{h} - (f')^2 h \right)^{\frac{3}{2}} = 0. \tag{3}$$

To solve $f(r)$, we set the substitution $\sin(\eta(r)) = f'(r)h(r)$, then Eq. (3) becomes

$$(\tan \eta)' + \left(\frac{2}{r} + \frac{h'}{2h} \right) \tan \eta - 3H \left(\frac{1}{h^{\frac{1}{2}}} \right) = 0 \Rightarrow \tan \eta = \frac{1}{\sqrt{h(r)}} \left(Hr - \frac{c}{r^2} \right),$$

where c is a constant of integration.[1]

Next, we write

$$l(r; H, c) = \frac{1}{\sqrt{h(r)}} \left(Hr - \frac{c}{r^2} \right) = \tan \eta$$

for convenience. Since $\sin \eta = f'h$ implies

$$\tan \eta = \frac{f'h}{\sqrt{1 - (f'h)^2}},$$

we get

$$f' = \frac{l}{h\sqrt{1 + l^2}},$$

so

$$f(r; H, c, \bar{c}) = \int_{r_{\text{ini}}}^{r} \frac{l(x; H, c)}{h(x)\sqrt{1 + l^2(x; H, c)}} \, dx + \bar{c},$$

where $\bar{c}$ is a constant and r_{ini} is an initial number. The solution is defined on the connected coordinates region I or region III. □

Next, we will discuss asymptotic behaviours of SSCMC hypersurfaces.

PROPOSITION 3.2. *For an SSCMC hypersurface* $\Sigma : (t = f(r), r, \theta, \phi)$ *mapping to region* I *of the maximally extended RN spacetime, the following results hold:*

(A) *If* $H > 0$, *then* $\lim_{r \to \infty} f'(r) = 1$ *and* Σ *is asymptotically null.*

[1]We multiply the integrating factor $r^2 h^{\frac{1}{2}}$ and get $(r^2 h^{\frac{1}{2}} \tan \eta)' = 3Hr^2$, so $r^2 h^{\frac{1}{2}} \tan \eta = Hr^3 - c$. Here we choose the constant of integration $-c$ in order to have a better expression when discussing the smoothness of SSCMC hypersurfaces.

(B) *If* $H = 0$, *then* $\lim_{r\to\infty} f'(r) = 0$ *and* Σ *is asymptotically spacelike.*

(C) *If* $H < 0$, *then* $\lim_{r\to\infty} f'(r) = -1$ *and* Σ *is asymptotically null.*

Proof: We observe the leading order of $f'(r)$,

$$\begin{aligned} f'(r) &= \frac{l(r;H,c)}{h(r)\sqrt{1+l^2(r;H,c)}} \\ &= \frac{Hr - c/r^2}{(1-2m/r+e^2/r^2)\sqrt{1-2m/r+e^2/r^2+\left(Hr-c/r^2\right)^2}}. \end{aligned}$$

When computing the square of the length of the timelike normal vector

$$\langle \nabla F, \nabla F\rangle = -\frac{1}{h(r)} + h(r)(f'(r))^2 = -\frac{1}{1-2m/r+e^2/r^2+(Hr-c/r^2)^2},$$

we know that $\lim_{r\to\infty}\langle \nabla F, \nabla F\rangle = 0$ if $H \neq 0$, and $\lim_{r\to\infty}\langle \nabla F, \nabla F\rangle = -1$ if $H = 0$. Hence Σ is asymptotically null for $H \neq 0$ and asymptotically spacelike for $H = 0$. □

PROPOSITION 3.3. *For an SSCMC hypersurface* $\Sigma : (t = f(r), r, \theta, \phi)$ *mapping to region* I *of the extended RN spacetime, the following conclusions hold:*

(A) *If* $c > r_+^3 H$, *then* $f'(r) < 0$ *near* $r = r_+$ *and* $\lim_{r\to(r_+)^+} f(r) = \infty$.

(B) *If* $c = r_+^3 H$, *then* $H \cdot f'(r) \geq 0$ *near* $r = r_+$ *and* $\lim_{r\to(r_+)^+} f(r)$ *is finite.*

(C) *If* $c < r_+^3 H$, *then* $f'(r) > 0$ *near* $r = r_+$ *and* $\lim_{r\to(r_+)^+} f(r) = -\infty$.

Furthermore, the spacelike condition of Σ *is preserved as* $r \to (r_+)^+$ *for all* $c \in \mathbb{R}$.

Proof: From the formula

$$\begin{aligned} f'(r) &= \frac{Hr - c/r^2}{h(r)\sqrt{h(r)+(Hr-c/r^2)^2}} \\ &= \frac{(Hr^3-c)r^2}{(r-r_+)(r-r_-)\sqrt{r^2(r-r_+)(r-r_-)+(Hr^3-c)^2}}, \end{aligned}$$

since Σ maps to region I, the denominator is always positive, so the sign of $f'(r)$ is determined by the sign of $Hr^3 - c$.

(A) If $c > r_+^3 H$, $f'(r) \sim O((r-r_+)^{-1})$ and $f'(r) < 0$ near r_+, so $\lim_{r\to(r_+)^+} f(r) = \infty$.

(B) If $c = r_+^3 H$, then

$$f'(r;H,c) = \frac{H(r-r_+)(r^2+rr_++r_+^2)r^2}{(r-r_+)(r-r_-)\sqrt{r^2(r-r_+)(r-r_-)+(r-r_+)^2(r^2+rr_++r_+^2)^2}}.$$

Since $f'(r) \sim O((r-r_+)^{-\frac{1}{2}})$ if $H \neq 0$ and the sign of $f'(r)$ is the same as the sign of H, so $\lim_{r\to(r_+)^+} f(r)$ is finite. If $H=0$, then $f'(r)=0$ and $f(r)=\bar{c}$ is a constant.

(C) If $c<r_+^3H$, $f'(r)\sim O((r-r_+)^{-1})$ and $f'(r)>0$ near r_+, so $\lim_{r\to(r_+)^+} f(r)=-\infty$.

Next, we look at

$$\langle \nabla F, \nabla F\rangle = -\frac{1}{1-2m/r+e^2/r^2+(Hr-c/r^2)^2} = -\frac{r^4}{r^2(r-r_+)(r-r_-)+(Hr^3-c)^2}.$$

If $c \neq r_+^3H$, then

$$\lim_{r\to(r_+)^+}\langle \nabla F, \nabla F\rangle = -\frac{r_+^4}{(Hr_+^3-c)^2} < 0,$$

so the hypersurface is spacelike. If $c=r_+^3H$, then

$$\lim_{r\to(r_+)^+}\langle \nabla F, \nabla F\rangle = \lim_{r\to(r_+)^+} -\frac{r^4}{r^2(r-r_+)(r-r_-)+H^2(r-r_+)^2(r^2+r_+r+r_+^2)^2} = -\infty,$$

so the hypersurface is spacelike as well. □

PROPOSITION 3.4. *For an SSCMC hypersurface $\Sigma : (t=f(r), r, \theta, \phi)$ mapping to region III of the extended RN spacetime, the following conclusions hold:*

(A) *If $c > r_-^3H$, then $f'(r) < 0$ near $r=r_-$ and $\lim_{r\to(r_-)^-} f(r) = -\infty$.*

(B) *If $c = r_-^3H$, then $H\cdot f'(r) \leq 0$ near $r=r_-$ and $\lim_{r\to(r_-)^-} f(r)$ is finite.*

(C) *If $c < r_-^3H$, then $f'(r) > 0$ near $r=r_-$ and $\lim_{r\to(r_-)^-} f(r) = \infty$.*

Furthermore, the spacelike condition of Σ is preserved as $r\to(r_-)^-$ for all $c\in\mathbb{R}$.

Proof: From the formula

$$f'(r) = \frac{Hr-c/r^2}{h(r)\sqrt{h(r)+(Hr-c/r^2)^2}} = \frac{(Hr^3-c)r^2}{(r-r_+)(r-r_-)\sqrt{r^2(r-r_+)(r-r_-)+(Hr^3-c)^2}},$$

since Σ maps to the region III, we know that $(r-r_+)(r-r_-)>0$, so the sign of $f'(r)$ is determined by the sign of Hr^3-c.

(A) If $c>r_-^3H$, $f'(r)\sim O((r-r_-)^{-1})$ and $f'(r)<0$ near r_-, so $\lim_{r\to(r_-)^-} f(r) = -\infty$.

(B) If $c=r_-^3H$, then

$$f'(r;H,c) = \frac{H(r-r_-)(r^2+rr_-+r_-^2)r^2}{(r-r_+)(r-r_-)\sqrt{r^2(r-r_+)(r-r_-)+(r-r_-)^2(r^2+rr_-+r_-^2)^2}},$$

$f'(r) \sim O((r-r_-)^{-\frac{1}{2}})$ if $H \neq 0$ and the sign of $f'(r)$ is the same as the sign of $-H$, so $\lim_{r\to(r_-)^-} f(r)$ is finite. If $H=0$, then $f'(r)=0$, and $f(r)=\bar{c}$ is a constant.

(C) If $c < r_-^3 H$, $f'(r) \sim O((r-r_-)^{-1})$ and $f'(r)>0$ near r_-, so $\lim_{r\to(r_-)^-} f(r) = \infty$.

Next, we compute

$$\langle \nabla F, \nabla F\rangle = -\frac{1}{1-2m/r+e^2/r^2+(Hr-c/r^2)^2} = -\frac{r^4}{r^2(r-r_+)(r-r_-)+(Hr^3-c)^2}.$$

If $c \neq r_-^3 H$, then

$$\lim_{r\to(r_-)^+} \langle \nabla F, \nabla F\rangle = -\frac{r_-^4}{(Hr_-^3-c)^2} < 0,$$

so the hypersurface is spacelike. If $c = r_-^3 H$, then

$$\lim_{r\to(r_-)^+} \langle \nabla F, \nabla F\rangle = \lim_{r\to(r_-)^+} -\frac{r^4}{r^2(r-r_+)(r-r_-)+H^2(r-r_-)^2(r^2+r_-r+r_-^2)^2} = -\infty$$

also implies that the hypersurface is spacelike. □

PROPOSITION 3.5. *For every $H \in \mathbb{R}$, $\lim_{r\to 0^+} f'(r)$ is finite for all $c \in \mathbb{R}$ so that all SSCMC hypersurfaces in region III will touch the spacetime singularity $r=0$.*

Proof: We observe the leading order of $f'(r)$,

$$f'(r) = \frac{l(r;H,c)}{h(r)\sqrt{1+l^2(r;H,c)}} = \frac{Hr^3-c}{(r^2-2mr+e^2)\sqrt{r^2(r^2-2mr+e^2)+(Hr^3-c)^2}}.$$

If $c \neq 0$, then

$$\lim_{r\to 0^+} f'(r) = -\frac{c}{e^2\sqrt{(-c)^2}} = -\frac{c}{e^2|c|} = -\frac{1}{e^2}\cdot \operatorname{sgn}(c)$$

is finite. If $c=0$, then

$$\lim_{r\to 0^+} f'(r) = \lim_{r\to 0^+} \frac{Hr^2}{(r^2-2mr+e^2)\sqrt{r^2-2mr+e^2+H^2r^4}} = 0.$$ □

Fig. 2 illustrates the correspondence of SSCMC hypersurfaces between standard coordinates (t, r, θ, ϕ) and (T, X, θ, ϕ).

3.2. SSCMC hypersurfaces in region I' and region III'

Recall the construction of the maximally extended RN spacetime. Since regions I' and III' come from second family of RN spacetime combining together with first family of RN spacetime in upside-down way, ∂_t direction points past direction. It implies that an SSCMC hypersurface of the form $(t = f(r), r, \theta, \phi)$ has opposite

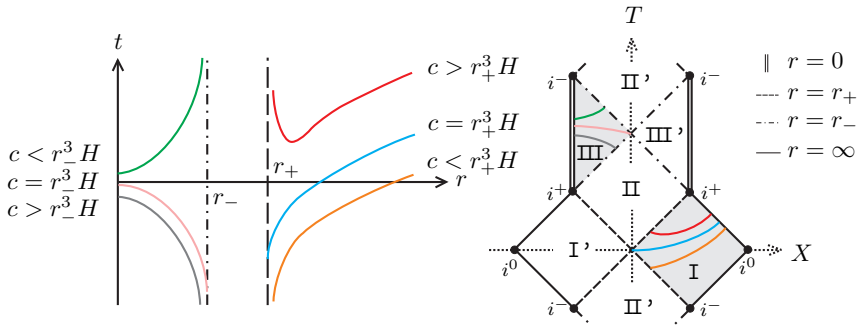


Fig. 2. SSCMC hypersurfaces with $H > 0$ in region I and III.

sign of the mean curvature when it maps to prime regions or nonprime regions. Therefore, we can get SSCMC hypersurfaces in region I' and III' by changing the sign of the mean curvature in reigon I and III.

Here we summarize SSCMC solutions in regions I' and III'. The constant mean curvature equation of an SSCMC hypersurface $\Sigma : (t = f(r), r, \theta, \phi)$ which maps to region I' or region III' of the maximally extended RN spacetime is

$$f'' + \left(\left(\frac{1}{h} - (f')^2 h\right)\left(\frac{2h}{r} + \frac{h'}{2}\right) + \frac{h'}{h}\right) f' + 3H\left(\frac{1}{h} - (f')^2 h\right)^{\frac{3}{2}} = 0.$$

The solution is

$$f'(r; H, c) = \frac{l(r; H, c)}{h(r)\sqrt{1 + l^2(r; H, c)}},$$

where

$$l(r; H, c) = \frac{1}{\sqrt{h(r)}}\left(-Hr + \frac{c}{r^2}\right)$$

and

$$f(r; H, c, \bar{c}) = \int_{r_{\text{ini}}}^{r} \frac{l(x; H, c)}{h(x)\sqrt{1 + l^2(x; H, c)}}\, dx + \bar{c},$$

where c and $\bar{c}$ are constants, and r_{ini} is an initial number.

Next, we can conclude and summarize asymptotic behaviours of SSCMC hypersurfaces in regions I' and III'.

PROPOSITION 3.6. *For an SSCMC hypersurface $\Sigma : (t = f(r), r, \theta, \phi)$ mapping to region I' of the maximally extended RN spacetime, the following results hold:*

(A) *If $H > 0$, then $\lim_{r\to\infty} f'(r) = -1$ and Σ is asymptotically null.*
(B) *If $H = 0$, then $\lim_{r\to\infty} f'(r) = 0$ and Σ is asymptotically spacelike.*
(C) *If $H < 0$, then $\lim_{r\to\infty} f'(r) = 1$ and Σ is asymptotically null.*

PROPOSITION 3.7. *For an SSCMC hypersurface* $\Sigma : (t = f(r), r, \theta, \phi)$ *mapping to region* I' *of the maximally extended RN spacetime, the following conclusions hold:*

(A) *If* $c > r_+^3 H$, *then* $f'(r) > 0$ *near* $r = r_+$ *and* $\lim_{r \to (r_+)^+} f(r) = -\infty$.

(B) *If* $c = r_+^3 H$, *then* $H \cdot f'(r) \leq 0$ *near* $r = r_+$ *and* $\lim_{r \to (r_+)^+} f(r)$ *is finite.*

(C) *If* $c < r_+^3 H$, *then* $f'(r) < 0$ *near* $r = r_+$ *and* $\lim_{r \to (r_+)^+} f(r) = \infty$.

Furthermore, the spacelike condition of Σ *is preserved as* $r \to (r_+)^+$ *for all* $c \in \mathbb{R}$.

PROPOSITION 3.8. *For an SSCMC hypersurface* $\Sigma : (t = f(r), r, \theta, \phi)$ *mapping to region* III' *of the maximally extended RN spacetime, the following conclusions hold:*

(A) *If* $c > r_-^3 H$, *then* $f'(r) > 0$ *near* $r = r_-$ *and* $\lim_{r \to (r_-)^-} f(r) = \infty$.

(B) *If* $c = r_-^3 H$, *then* $H \cdot f'(r) \geq 0$ *near* $r = r_-$ *and* $\lim_{r \to (r_-)^-} f(r)$ *is finite.*

(C) *If* $c < r_-^3 H$, *then* $f'(r) < 0$ *near* $r = r_-$ *and* $\lim_{r \to (r_-)^-} f(r) = -\infty$.

Furthermore, the spacelike condition of Σ *is preserved as* $r \to (r_-)^-$ *for all* $c \in \mathbb{R}$.

PROPOSITION 3.9. *For every* $H \in \mathbb{R}$, $\lim_{r \to 0^+} f'(r)$ *is finite for all* $c \in \mathbb{R}$ *so that all SSCMC hypersurfaces in region* III' *will touch the spacetime singularity* $r = 0$.

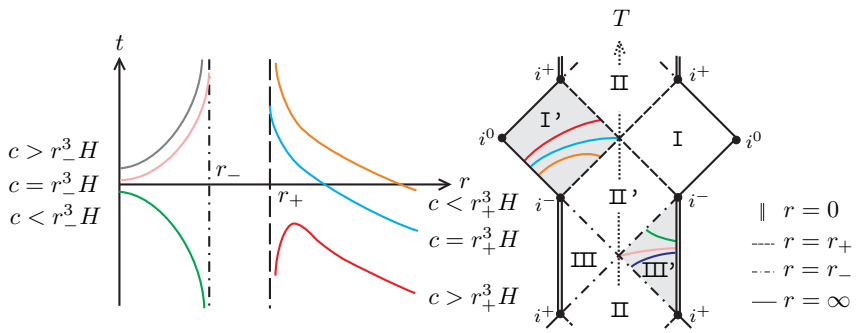


Fig. 3. SSCMC hypersurfaces with $H > 0$ in regions I' and III'.

Fig. 3 illustrates SSCMC hypersurfaces between standard coordinates (t, r, θ, ϕ) and (T, X, θ, ϕ) in regions I' and III'.

3.3. Cylindrical hypersurfaces in region II

In the RN spacetime region II, because $h(r) = 1 - 2m/r + e^2/r^2 < 0$, we know that $-\partial_r$ is future directed timelike and ∂_t is spacelike. To find SSCMC hypersurfaces

in region II, we may first assume that Σ is of the form $(t, r = g(t), \theta, \phi)$ for some function $r = g(t)$.

First of all, we can get cylindrical hypersurfaces. This result is known in [15], so we skip the proof here. We also refer to [9] for further explanations.

PROPOSITION 3.10. [15] *Each constant slice $r = r_0$, $r_0 \in (r_-, r_+)$ in region II is an SSCMC hypersurface with mean curvature*

$$H(r_0) = \frac{2r_0^2 - 3mr_0 + e^2}{3r_0^2\sqrt{-r_0^2 + 2mr_0 - e^2}}.$$

We say $r = r_0$, $r_0 \in (r_-, r_+)$ is a cylindrical hypersurface.

Next, we have to prove more properties on cylindrical hypersurfaces as the following ones.

PROPOSITION 3.11. *For all $m^2 > e^2 > 0$, the function*

$$H(r) = \frac{2r^2 - 3mr + e^2}{3r^2\sqrt{-r^2 + 2mr - e^2}}$$

is an increasing function on (r_-, r_+). Furthermore, cylindrical hypersurfaces $r = r_0, r_0 \in (r_-, r_+)$ have the following properties:

(A) *If $r_0 \in (r_-, (3m + \sqrt{9m^2 - 8e^2})/4)$, then $H(r_0) < 0$ and $\lim_{r \to (r_-)^+} H(r) = -\infty$.*

(B) *If $r_0 = (3m + \sqrt{9m^2 - 8e^2})/4$, then $H(r_0) = 0$ is a maximal hypersurface.*

(C) *If $r_0 \in ((3m + \sqrt{9m^2 - 8e^2})/4, r_+)$, then $H(r_0) > 0$ and $\lim_{r \to (r_+)^-} H(r) = \infty$.*

Proof: First of all, we will prove that $H(r)$ is increasing. Direct computation gives

$$H'(r) = \frac{2r^4 - 8mr^3 + 9m^2r^2 + 3e^2r^2 - 8e^2mr + 2e^4}{3r^3(-r^2 + 2mr - e^2)^{\frac{3}{2}}}.$$

We will show that $H'(r) > 0$ for all $m^2 > e^2 > 0$ on (r_-, r_+). Let

$$p(r) = 2r^4 - 8mr^3 + 9m^2r^2 + 3e^2r^2 - 8e^2mr + 2e^4$$

defined on $[r_-, r_+]$. It suffices to show that the absolute minimum value of $p(r)$ on $[r_-, r_+]$ is positive.

Since

$$p'(r) = 8r^3 - 24mr^2 + 18m^2r + 6e^2r - 8e^2m,$$

there are at most three critical points of $p(r)$ on $[r_-, r_+]$; that is, there are at most three real roots r_i, $i = 1, 2, 3$ satisfying $p'(r_i) = 0$ on $[r_-, r_+]$. Since

$$p(r) = p'(r)Q(r) + \frac{(m^2 - e^2)}{2}(-3r^2 + 9mr - 4e^2),$$

we have

$$p(r_i) = \frac{(m^2 - e^2)}{2}(-3r_i^2 + 9mr_i - 4e^2).$$

The polynomial

$$\tilde{p}(r) = -3r^2 + 9mr - 4e^2 = -3\left(r - \frac{3}{2}m\right)^2 + \frac{27}{4}m^2 - e^2$$

satisfies

$$\tilde{p}(r_\pm) = 3mr_\pm - e^2 = 3m^2 - e^2 \pm 3m\sqrt{m^2 - e^2} = \frac{e^2 + 3m^2e^2}{3m^2 - e^2 \mp 3m\sqrt{m^2 - e^2}} > 0,$$

so $\tilde{p}(r) > 0$ on (r_-, r_+). It implies that all critical points r_i in the interval (r_-, r_+) must satisfy $p(r_i) > 0$. In addition, we have $p(r_\pm) = 2m(m^2 - e^2)r_\pm + e^2(m^2 - e^2) > 0$. Thus, the absolute minimum value of $p(r)$ on $[r_-, r_+]$ must be positive, and hence $H'(r) > 0$.

Since $H(r)$ is increasing, $H(r) = 0$ has a unique solution

$$r = \frac{3m + \sqrt{9m^2 - 8e^2}}{4}$$

in the interval (r_-, r_+). Here we remark that

$$r = \frac{3m - \sqrt{9m^2 - 8e^2}}{4} < r_-.$$

Since

$$\lim_{r \to r_\pm} H(r) = \lim_{r \to r_\pm} \frac{2r^2 - 3mr + e^2}{3r^2\sqrt{-(r - r_+)(r - r_-)}} = \pm\infty,$$

all properties stated in Proposition 3.11 are characterized. □

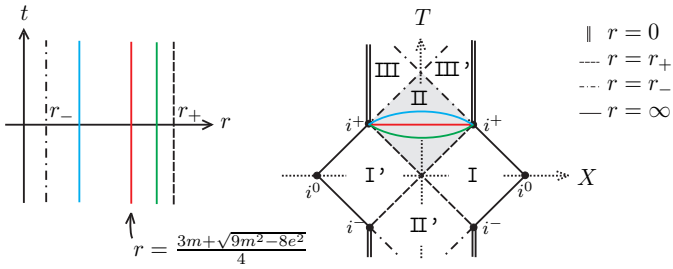


Fig. 4. Cylindrical hypersurfaces in region II.

From the above argument, we can plot cylindrical hypersurfaces in region II as Fig 4.

3.4. Noncylindrical SSCMC hypersurfaces in region II

For $r = g(t) \neq$ constant, we piecewisely consider its inverse function $t = f(r)$ with $f'(r) \neq 0$ whenever it is defined. Here we allow $f'(r) = \infty$ or $-\infty$ because they correspond to $g'(t) = 0$ at some point.

PROPOSITION 3.12. *Suppose that $\Sigma : (t = f(r), r, \theta, \phi)$ is an SSCMC hypersurface in the RN spacetime which maps to the region II. Then*

$$f'(r) = \begin{cases} \dfrac{1}{-h(r)}\sqrt{\dfrac{l^2(r;H,c)}{l^2(r;H,c)-1}} & \text{if } f' > 0, \\ \dfrac{1}{h(r)}\sqrt{\dfrac{l^2(r;H,c)}{l^2(r;H,c)-1}} & \text{if } f' < 0, \end{cases} \quad \text{where } l(r;H,c) = \frac{1}{\sqrt{-h(r)}}\left(-Hr + \frac{c}{r^2}\right).$$

The integration of $f'(r)$ gives

$$f(r;H,c,\bar{c}) = \int_{r_{\text{ini}}}^{r} \frac{1}{-h(x)}\sqrt{\frac{l^2(x;H,c)}{l^2(x;H,c)-1}}\,dx + \bar{c}, \qquad \text{or} \tag{4}$$

$$f(r;H,c,\bar{c}) = \int_{r_{\text{ini}}}^{r} \frac{1}{h(x)}\sqrt{\frac{l^2(x;H,c)}{l^2(x;H,c)-1}}\,dx + \bar{c}, \tag{5}$$

according to the sign of $f'(r)$, where r_{ini} is an initial number, and $c, \bar{c}$ are two constants.

The proof of Proposition 3.12 is similar to the proof of Proposition 3.1. We refer to the note [9] for the complete proof of this proposition.

3.5. Cylindrical hypersurfaces in region II'

PROPOSITION 3.13. [15] *Each constant slice $r = r_0$, $r_0 \in (r_-, r_+)$ is an SSCMC hypersurface (called cylindrical hypersurface) with mean curvature*

$$H(r_0) = -\frac{2r_0^2 - 3mr_0 + e^2}{3r_0^2\sqrt{-r_0^2 + 2mr_0 - e^2}}.$$

PROPOSITION 3.14. *For all $m^2 > e^2 > 0$, the function*

$$H(r) = \frac{2r^2 - 3mr + e^2}{3r^2\sqrt{-r^2 + 2mr - e^2}}$$

is a decreasing function on (r_-, r_+). Furthermore, cylindrical hypersurfaces $r = r_0, r_0 \in (r_-, r_+)$ have the following properties:

(A) *If $r_0 \in (r_-, (3m + \sqrt{9m^2 - 8e^2})/4)$, then $H(r_0) > 0$ and $\lim_{r \to (r_-)^+} H(r) = \infty$.*

(B) *If* $r_0 = (3m + \sqrt{9m^2 - 8e^2})/4$, *then* $H(r_0) = 0$ *is a maximal hypersurface.*
(C) *If* $r_0 \in ((3m + \sqrt{9m^2 - 8e^2})/4, r_+)$, *then* $H(r_0) < 0$ *and* $\lim_{r \to (r_+)^-} H(r) = -\infty$.

3.6. Noncylindrical SSCMC hypersurfaces in region II'

For $r = g(t) \neq$ constant, we piecewisely consider its inverse function $t = f(r)$ with $f'(r) \neq 0$ whenever it is defined. Here we allow $f'(r) = \infty$ or $-\infty$ because they correspond to $g'(t) = 0$ at some point.

PROPOSITION 3.15. *Suppose that* $\Sigma : (t = f(r), r, \theta, \phi)$ *is an SSCMC hypersurface in the RN spacetime which maps to the region* II'. *Then*

$$f'(r) = \begin{cases} \dfrac{1}{-h(r)}\sqrt{\dfrac{l^2(r;H,c)}{l^2(r;H,c)-1}} & \text{if } f' > 0, \\[2ex] \dfrac{1}{h(r)}\sqrt{\dfrac{l^2(r;H,c)}{l^2(r;H,c)-1}} & \text{if } f' < 0, \end{cases} \qquad \text{where } l(r;H,c) = \frac{1}{\sqrt{-h(r)}}\left(Hr - \frac{c}{r^2}\right).$$

The integration of $f'(r)$ *gives*

$$f(r;H,c,\bar{c}) = \int_{r_{\text{ini}}}^{r} \frac{1}{-h(x)}\sqrt{\frac{l^2(x;H,c)}{l^2(x;H,c)-1}}\,dx + \bar{c}, \qquad \text{or} \tag{6}$$

$$f(r;H,c,\bar{c}) = \int_{r_{\text{ini}}}^{r} \frac{1}{h(x)}\sqrt{\frac{l^2(x;H,c)}{l^2(x;H,c)-1}}\,dx + \bar{c}, \tag{7}$$

according to the sign of $f'(r)$, *where* r_{ini} *is an initial number, and* $c, \bar{c}$ *are two constants.*

3.7. Position of the SSCMC hypersurfaces in region II and region II'

Spacelike condition of an SSCMC hypersurface in region II in fact restricts the domain of $f(r)$ in region II; that is, from the formula

$$l(r;H,c) = \frac{1}{\sqrt{-h(r)}}\left(-Hr + \frac{c}{r^2}\right) > 1 \Leftrightarrow c > Hr^3 + r(-r^2 + 2mr - e^2)^{\frac{1}{2}},$$

we consider a function $k(r;H) = Hr^3 + r(-r^2 + 2mr - e^2)^{\frac{1}{2}}$ defined on (r_-, r_+), then the domain of $f(r)$ in region II will be

$$\{r \in (r_-, r_+) \mid k(r;H) < c\} \cup \{r \in (r_-, r_+) \mid k(r;H) = c \text{ and } f(r) \text{ is finite}\}.$$

At first, we can describe the function $k(r;H)$ as follows.

PROPOSITION 3.16. *Fixed* $H \in \mathbb{R}$, *the function* $k(r;H)$ *has a unique maximum point at* $r = R_H$, *where* $r = R_H$ *is a cylindrical hypersurface with mean curvature* H.

Proof: Direct computation gives (the prime means to take derivative with respect to r)

$$\begin{aligned} k'(r;H) &= 3Hr^2 + (-r^2+2mr-e^2)^{\frac{1}{2}} + r\cdot\frac{1}{2}(-r^2+2mr-e^2)^{-\frac{1}{2}}\cdot(-2r+2m) \\ &= 3r^2(H-H(r)). \end{aligned}$$

From Proposition 3.11, we know that $H(r)$ is an increasing function on (r_-, r_+), so $k(r;H)$ has only one critical point $r = R_H$, where $H(R_H) = H$. Furthermore, $k'(r;H) > 0$ on (r_-, R_H) and $k'(r;H) < 0$ on (R_H, r_+), so the critical point will attain the maximum value of $k(r;H)$. □

Similarly, spacelike condition of an SSCMC hypersurface in region II' in fact restricts the domain of $f(r)$ in region II'; that is, from

$$l(r;H,c) = \frac{1}{\sqrt{-h(r)}}\left(Hr - \frac{c}{r^2}\right) > 1 \Leftrightarrow c < Hr^3 - r(-r^2+2mr-e^2)^{\frac{1}{2}},$$

we consider another function $\tilde{k}(r;H) = Hr^3 - r(-r^2+2mr-e^2)^{\frac{1}{2}}$ defined on (r_-, r_+), then the domain of $f(r)$ in region II' is

$$\{r \in (r_-, r_+) \mid \tilde{k}(r;H) > c\} \cup \{r \in (r_-, r_+) \mid \tilde{k}(r;H) = c \text{ and } f(r) \text{ is finite}\}.$$

PROPOSITION 3.17. *For a fixed $H \in \mathbb{R}$, the function $\tilde{k}(r;H)$ has a unique minimum point at $r = r_H$, where $r = r_H$ is a cylindrical hypersurface with mean curvature H.*

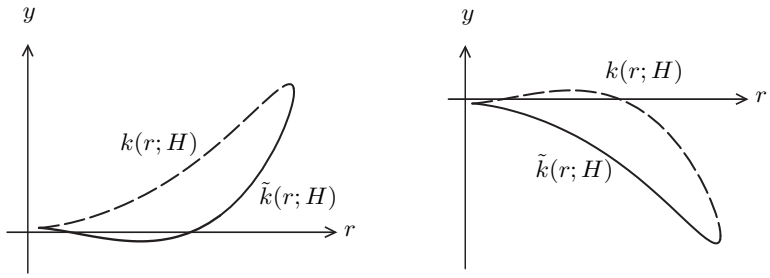
Proof: Direct computation gives

$$\begin{aligned} \tilde{k}'(r;H) &= 3Hr^2 - (-r^2+2mr-e^2)^{\frac{1}{2}} - r\cdot\frac{1}{2}(-r^2+2mr-e^2)^{-\frac{1}{2}}\cdot(-2r+2m) \\ &= 3r^2(H-H(r)). \end{aligned}$$

From Proposition 3.13, we know that $H(r)$ is a decreasing function on (r_-, r_+), so $\tilde{k}(r;H)$ has only one critical point $r = r_H$, where $H(r_H) = H$. Furthermore, $\tilde{k}'(r;H) < 0$ on (r_-, r_H) and $\tilde{k}'(r;H) > 0$ on (r_H, r_+), so the critical point will attain the minimum value of $\tilde{k}(r;H)$. □

For a fixed $H \in \mathbb{R}$, we plot graphs of $k(r;H)$ and $\tilde{k}(r;H)$ in Fig. 5. Two graphs of functions $(r, y = k(r;H))$ and $(r, y = \tilde{k}(r;H))$ form a closed loop. From this loop and a horizontal line $y = c$, it is much easier to know the domain of the function $f(r)$. That is, the preimage of the line outside the loop will determine the domain of $f(r)$ and hence we know the position of the SSCMC hypersurface.

More precisely, the following proposition will describe SSCMC hypersurfaces in regions II or II'.



(a) $H > 0$ (b) $H < 0$

Fig. 5. Graphs of $k(r; H)$ and $\tilde{k}(r; H)$.

PROPOSITION 3.18. *Given $H > 0$, denote $C_H = \max_{r\in(r_-,r_+)} k(r; H) = k(R_H; H)$, where $k(r; H) = Hr^3 + r(-r^2 + 2mr - e^2)^{\frac{1}{2}}$, and $c_H = \min_{r\in(r_-,r_+)} \tilde{k}(r; H) = \tilde{k}(r_H; H)$, where $\tilde{k}(r; H) = Hr^3 - r(-r^2 + 2mr - e^2)^{\frac{1}{2}}$. There are seven types of noncylindrical SSCMC hypersurfaces in region* II *or region* II' *with formulae of $f(r)$ in Propositions* 3.12 *and* 3.15 *according to the value of c:*

(A) *If $c > C_H$, then $f(r)$ is defined on (r_-, r_+) in region* II.
(B) *If $c = C_H$, then $f(r)$ is defined on (r_-, R_H) or (R_H, r_+) in region* II.
(C) *If $r_+^3 H < c < C_H$, then $f(r)$ is defined on $(r_-, r']$ or $[r'', r_+)$ in region* II, *where $k(r') = k(r'') = c$. At $r = r'$ or $r = r''$, we can take another function also defined on $(r_-, r']$ or $[r'', r_+)$ but different sign of slope joined at $r = r'$ or $r = r''$, respectively, such that the union of two graphs of functions forms a smooth SSCMC hypersurface in region* II.
(D) *If $r_-^3 H < c < r_+^3 H$, then $f(r)$ is defined on $(r_-, r']$ in region* II, *or $[r'', r_+)$ in region* II', *where $k(r') = \tilde{k}(r'') = c$. At $r = r'$ or $r = r''$, we can take another function also defined on $(r_-, r']$ or $[r'', r_+)$ but different sign of slope joined at $r = r'$ or $r = r''$, respectively, such that the union of two graphs of functions forms a smooth SSCMC hypersurface in region* II *or* II', *respectively.*
(E) *If $c_H < c < r_-^3 H$, then $f(r)$ is defined on $(r_-, r']$ or $[r'', r_+)$ in region* II', *where $\tilde{k}(r') = \tilde{k}(r'') = c$. At $r = r'$ or $r = r''$, we can take another function also defined on $(r_-, r']$ or $[r'', r_+)$ but different sign of slope joined at $r = r'$ or $r = r''$, respectively, such that the union of two graphs of functions forms a smooth SSCMC hypersurface in region* II'.
(F) *If $c = c_H$, then $f(r)$ is defined on (r_-, r_H) or (r_H, r_+) in region* II'.
(G) *If $c < c_H$, then $f(r)$ is defined on (r_-, r_+) in region* II'.

Before proving Proposition 3.18, we remark that for $H = 0$ or $H < 0$, SSCMC hypersurfaces in regions II or II' can be treated similarly. The only difference is

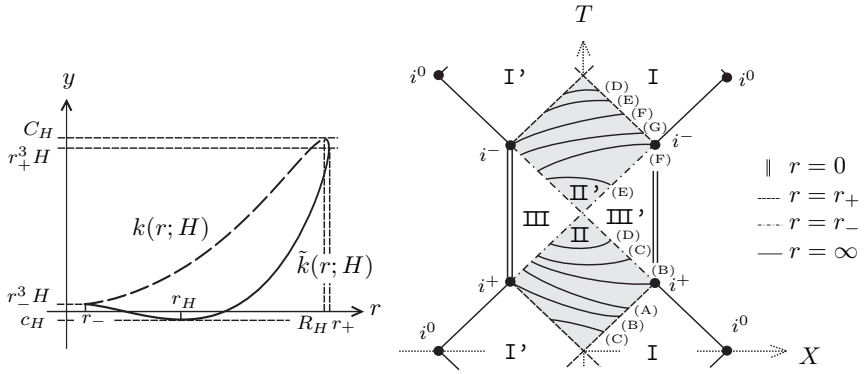


Fig. 6. Using $y = c$, $k(r; H)$ and $\tilde{k}(r; H)$ to characterize different types of SSCMC hypersurfaces. This figure illustrates the $H > 0$ case.

that different types of noncylindrical SSCMC hypersurfaces are characterized by the values $C_H > r_-^3 H = r_+^3 H > c_H$ or $C_H > r_-^3 H > r_+^3 H > c_H$, respectively. All types of noncylindrical SSCMC hypersurfaces can be similarly interpreted from the graphs of $k(r; H)$ and $\tilde{k}(r; H)$, Fig. 5 (b) for example.

Proof: The key point is to make sure the order of $f'(r)$ when $l(r; H, c) \to 1$; that is, we observe the formula

$$|f'(r)| = \frac{1}{-h(r)}\sqrt{\frac{l^2(r; H, c)}{l^2(r; H, c) - 1}}.$$

We only need to take care of the denominator part $\frac{1}{\sqrt{l^2(r;H,c)-1}}$. For the cases (A) or (G), since $l(r; H, c) > 1$ for all $r \in (r_-, r_+)$, SSCMC hypersurfaces will range over (r_-, r_+) in the region II or II', respectively.

For cases (B) or (F), since it corresponds to the maximum value of $k(r; H)$ or minimum value of $\tilde{k}(r; H)$, solutions of $l^2(r; H, c) - 1 = 0$ are double real roots at $r = R_H$ or $r = r_H$. It implies that $|f'(r)|$ is of order $O(|r - R_H|^{-1})$ or $O(|r - r_H|^{-1})$, so we get $|f(r)| \to \infty$ as $r \to R_H$ or $r \to r_H$.

For cases (C), (D), or (E), since solutions of $l^2(r; H, c) - 1 = 0$ will be two distinct real roots, say $r = r'$ or $r = r''$, we know that $|f'(r)|$ is of order $O(|r - r'|^{-\frac{1}{2}})$ or $O(|r - r''|^{-\frac{1}{2}})$. It indicates that $f(r)$ is finite at $r = r'$ or $r = r''$. We choose one function with $f'(r) > 0$ and another function with $f'(r) < 0$ with the same domain, the same c, and adjust another constant $\bar{c}$ so that two functions $f(r)$ have the same value at $r = r'$ or $r = r''$. For example, from the formulae (4) and (5), or (6) and (7), if we take the initial number as $r_{\text{ini}} = r'$ or $r_{\text{ini}} = r''$, then they share the same $\bar{c}$ value. Therefore, the union of two graphs of functions forms a continuous SSCMC hypersurface in region II or region II'.

We still need to prove the smoothness at the joint point of two SSCMC hypersurfaces. Here we look at the case $r = r'$, and the case $r = r''$ is similar. We consider their inverse functions of $t = f(r)$ in (4), and (5), or (6) and (7), with $r_{\text{ini}} = r'$ that is, we rewrite the SSCMC hypersurface as a graph of $r = g(t)$ such that $g(\bar{c}) = r'$. Direct computation (by induction) gives

$$g^{(2k+1)}(t) = \begin{cases} \sum_{i=0}^{k} A_{k,i}(l^2-1)^{i+\frac{1}{2}} & \text{if } t < \bar{c}, \\ (-1)^{2k+1}\sum_{i=0}^{k} A_{k,i}(l^2-1)^{i+\frac{1}{2}} & \text{if } t > \bar{c}, \end{cases}$$

and

$$g^{(2k)}(t) = \begin{cases} \sum_{i=0}^{k} B_{k,i}(l^2-1)^{i} & \text{if } t < \bar{c}, \\ (-1)^{2k+1}\sum_{i=0}^{k} B_{k,i}(l^2-1)^{i} & \text{if } t > \bar{c}, \end{cases}$$

where $A_{k,i}$ and $B_{k,i}$ are functions of h, l and their derivatives with respective to r. As $t \to \bar{c}$, we have $r \to r'$ and $\lim_{r\to r'} l^2 - 1 = 0$, and it implies

$$\lim_{t\to\bar{c}^-} g^{(2k+1)}(t) = \lim_{t\to\bar{c}^+} g^{(2k+1)}(t) = 0 \qquad \text{and} \qquad \lim_{t\to\bar{c}^-} g^{(2k)}(t) = \lim_{t\to\bar{c}^+} g^{(2k)}(t) = B_{k,0}.$$

Hence the union of two SSCMC hypersurfaces is smooth at the joint point. □

PROPOSITION 3.19. *Suppose that $f(r)$ is a solution of SSCMC equation defined near $r = r_-$ or $r = r_+$. Then $\lim_{r\to(r_-)^+} |f'(r)| = \infty$ or $\lim_{r\to(r_+)^-} |f'(r)| = \infty$. Moreover, spacelike condition of the SSCMC hypersurface $(t = f(r), r, \theta, \phi)$ still holds near the coordinate singularities.*

Proof: From the formula

$$|f'(r)| = \frac{1}{-h(r)}\sqrt{\frac{l^2(r;H,c)}{l^2(r;H,c)-1}} = \frac{1}{|r-r_-||r-r_+|}\sqrt{\frac{l^2(r;H,c)}{l^2(r;H,c)-1}},$$

we know that $\lim_{r\to(r_\mp)^\pm} |f'(r)| = \infty$. Spacelike property can be extended at $r = r_\pm$ because

$$\lim_{r\to r_\pm} \langle \nabla F, \nabla F\rangle = \lim_{r\to r_\pm} \frac{1}{h(l^2-1)} = -\frac{1}{\left(-Hr_\pm + c/r_\pm^2\right)^2} < 0. \qquad \square$$

4. Characterization of SSCMC hypersurfaces in the extended RN spacetime

From all discussions in Section 3, we are ready to prove the characterization theorem.

CHARACTERIZATION THEOREM. *All SSCMC hypersurfaces in the extended RN spacetime can be determined by two parameters c and $\bar{c}$. In other words, SSCMC*

hypersurfaces in each standard RN coordinates region are determined by two constants of integration c and $\bar{c}$, and we can take the same c value and adjust the value of $\bar{c}$ such that the union of SSCMC hypersurfaces in the extended RN spacetime is C^1 and thus C^∞ smooth.

Here we aim to describe the characterization theorem in more detail. In the following arguments, we add indices I, II, III, I', II', and III to each constant of integration to distinguish SSCMC hypersurfaces in different regions. First of all, we start form an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}}$ in region I to construct a maximally extended smooth SSCMC hypersurface in the extended RN spacetime. We divide the construction into three theorems according to the value $c_{\mathrm{I}} > r_+^3 H$, $c_{\mathrm{I}} = r_+^3 H$, or $c_{\mathrm{I}} < r_+^3 H$.

THEOREM 4.1. *Given constant mean curvature $H \in \mathbb{R}$, $c_{\mathrm{I}} > r_+^3 H$, and $\bar{c}_{\mathrm{I}} \in \mathbb{R}$, it will determine an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}}$ in region I. We can take an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ in region II with $c_{\mathrm{II}} = c_{\mathrm{I}}$ and with some $\bar{c}_{\mathrm{II}}$ determined by $\bar{c}_{\mathrm{I}}$ such that $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ is a smooth SSCMC hypesurface. Furthermore, suppose that $C_H = \max_{r\in(r_-,r_+)} Hr^3 + r(-r^2 + 2mr - e^2)^{\frac{1}{2}}$.*

(A) *If $c_{\mathrm{I}} > C_H$, then we can take an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{III}},\bar{c}_{\mathrm{III}}}$ in region III with $c_{\mathrm{III}} = c_{\mathrm{I}}$ and with some $\bar{c}_{\mathrm{III}}$ determined by $\bar{c}_{\mathrm{I}}$ such that $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}} \cup \Sigma_{H,c_{\mathrm{III}},\bar{c}_{\mathrm{III}}}$ is a smooth SSCMC hypersurface in the extended RN spacetime.*

(B) *If $c_{\mathrm{I}} = C_H$, then $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ is a smooth SSCMC hypersurface ranging from region I to II in the extended RN spacetime.*

(C) *If $r_+^3 H < c_{\mathrm{I}} < C_H$, then we can take an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{I}'},\bar{c}_{\mathrm{I}'}}$ in region I' with $c_{\mathrm{I}'} = c_{\mathrm{I}}$ and with some $\bar{c}_{\mathrm{I}'}$ determined by $\bar{c}_{\mathrm{I}}$ such that $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}} \cup \Sigma_{H,c_{\mathrm{I}'},\bar{c}_{\mathrm{I}'}}$ is a smooth SSCMC hypersurface in the extended RN spacetime.*

Proof: The SSCMC hypersurface $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}}$ in region I is $(t = f(r; H, c_{\mathrm{I}}, \bar{c}_{\mathrm{I}}), r, \theta, \phi)$ in the standard RN coordinates. Since $c_{\mathrm{I}} > r_+^3 H$, we know that $\lim_{r\to(r_+)^+} f(r) = +\infty$. It implies that $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}}$ will touch the interface of region I and II. Recall that an SSCMC solution $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}}$ in region I is

$$f_{\mathrm{I}}(r; H, c_{\mathrm{I}}, \bar{c}_{\mathrm{I}}) = \int_{r_{\mathrm{I}}}^{r} \frac{l(x; H, c_{\mathrm{I}})}{h(x)\sqrt{1 + l^2(x; H, c_{\mathrm{I}})}}\, dx + \bar{c}_{\mathrm{I}},$$

for some initial number r_{I} in region I, and an SSCMC solution $\Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ in region II with $\lim_{r\to(r_+)^-} f(r) = +\infty$ is

$$f_{\mathrm{II}}(r; H, c_{\mathrm{II}}, \bar{c}_{\mathrm{II}}) = \int_{r_{\mathrm{II}}}^{r} \frac{1}{-h(x)}\sqrt{\frac{l^2(x; H, c_{\mathrm{II}})}{l^2(x; H, c_{\mathrm{II}}) - 1}}\, dx + \bar{c}_{\mathrm{II}},$$

where r_{II} is another initial number in region II.

Notice that from Proposition 3.18 (A), (B), and (C), we know that $c_{\mathrm{II}} > r_+^3 H$. Here we will find conditions to guarantee that $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ is C^1 smooth at the joint point. Let $\bar{f}_{\mathrm{I}}'(r; H, c_{\mathrm{I}}) = 1/h(r) + f_{\mathrm{I}}'(r; H, c_{\mathrm{I}})$ and $\bar{f}_{\mathrm{II}}'(r; H, c_{\mathrm{II}}) = 1/h(r) + f_{\mathrm{II}}'(r; H, c_{\mathrm{II}})$ near $r = r_+$. By Taylor's expansion, since

$$\begin{aligned} f_{\mathrm{I}}' &= \frac{l}{h\sqrt{1+l^2}} = -\frac{1}{h}\sqrt{1-\frac{1}{1+l^2}} = -\frac{1}{h}\sum_{n=0}^{\infty}(-1)^n C_n^{\frac{1}{2}}\left(\frac{1}{1+l^2}\right)^n \\ &= -\frac{1}{h} + \frac{1}{2}\cdot\frac{1}{h+(Hr-c_{\mathrm{I}}/r^2)^2} + \frac{1}{8}\cdot\frac{h}{(h+(Hr-c_{\mathrm{I}}/r^2)^2)^2} + \cdots, \end{aligned}$$

we have

$$\lim_{r\to(r_+)^+} \bar{f}_{\mathrm{I}}' = \frac{1}{2\left(Hr_+ - c_{\mathrm{I}}/r_+^2\right)^2}.$$

Similarly, since

$$f_{\mathrm{II}}' = \frac{1}{-h}\sqrt{\frac{l^2}{l^2-1}} = -\frac{1}{h}+\frac{1}{2}\cdot\frac{1}{h+(-Hr+c_{\mathrm{II}}/r^2)^2}+\frac{1}{8}\cdot\frac{h}{(h+(-Hr+c_{\mathrm{II}}/r^2)^2)^2}+\cdots,$$

we have

$$\lim_{r\to(r_+)^-} \bar{f}_{\mathrm{II}}' = \frac{1}{2\left(-Hr_+ + c_{\mathrm{II}}/r_+^2\right)^2}.$$

The C^1 smoothness condition at $r = r_+$ requires

$$\lim_{r\to(r_+)^+} \bar{f}_{\mathrm{I}}' = \lim_{r\to(r_+)^-} \bar{f}_{\mathrm{II}}' \Rightarrow c_{\mathrm{II}} = c_{\mathrm{I}} \text{ or } c_{\mathrm{II}} = 2r_+^3 H - c_{\mathrm{I}}.$$

Since $c_{\mathrm{II}} > r_+^3 H$, we get that $c_{\mathrm{II}} = c_{\mathrm{I}}$ is the only choice.

Next, we will determine the relation between $\bar{c}_{\mathrm{II}}$ and $\bar{c}_{\mathrm{I}}$. Since

$$\lim_{r\to(r_+)^+} \tan(V) = \lim_{r\to(r_+)^-} \tan(V)^2,$$

it gives that

$$\begin{aligned} &\exp\left(\alpha\left(\int_{r_{\mathrm{I}}}^{r_+} \bar{f}_{\mathrm{I}}'(r)\,dr + r_{\mathrm{I}} + \frac{r_+^2}{r_+-r_-}\ln|r_{\mathrm{I}}-r_+| - \frac{r_-^2}{r_+-r_-}\ln|r_{\mathrm{I}}-r_-| + \bar{c}_{\mathrm{I}}\right)\right) \\ &= \exp\left(\alpha\left(\int_{r_{\mathrm{II}}}^{r_+} \bar{f}_{\mathrm{II}}'(r)\,dr + r_{\mathrm{II}} + \frac{r_+^2}{r_+-r_-}\ln|r_{\mathrm{II}}-r_+| - \frac{r_-^2}{r_+-r_-}\ln|r_{\mathrm{II}}-r_-| + \bar{c}_{\mathrm{II}}\right)\right). \end{aligned}$$

Since $c_{\mathrm{I}} = c_{\mathrm{II}}$, we know that the expressions for $\bar{f}_{\mathrm{I}}'(r)$ and $\bar{f}_{\mathrm{II}}'(r)$ are the same,

[2]Here we need to know some relations on the construction of the Penrose diagram of the Reissner–Nordström spacetime, and we refer to [9] for more discussions.

so we write them as $\bar{f}'(r)$. Therefore, we can take

$$\bar{c}_{\mathrm{II}} = \bar{c}_{\mathrm{I}} + \int_{r_{\mathrm{I}}}^{r_{\mathrm{II}}} \bar{f}'(r)\,dr + (r_{\mathrm{I}} - r_{\mathrm{II}}) + \frac{r_+^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{I}} - r_+}{r_{\mathrm{II}} - r_+}\right| - \frac{r_-^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{I}} - r_-}{r_{\mathrm{II}} - r_-}\right|$$

such that $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ is continuous at the joint point $r = r_+$.

From Proposition 3.18, there are three types of SSCMC hypersurfaces when $c_{\mathrm{I}} > r_+^3 H$.

(A) <u>If $c_{\mathrm{I}} > C_H$</u>, then the next joint point will be $r = r_-$ at the interface of regions II and III. The SSCMC hypersurfaces $\Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ and $\Sigma_{H,c_{\mathrm{III}},\bar{c}_{\mathrm{III}}}$ have the behaviour $\lim_{r\to(r_-)^+} f_{\mathrm{II}}(r) = \lim_{r\to(r_-)^-} f_{\mathrm{III}}(r) = -\infty$. Notice that from Proposition 3.4, we know that $c_{\mathrm{III}} > r_-^3 H$. We write the SSCMC solution as $f_{\mathrm{III}}(r; H, c_{\mathrm{III}}, \bar{c}_{\mathrm{III}})$, $\bar{f}'_{\mathrm{III}}(r; H, c_{\mathrm{III}}) = 1/h(r) + f'_{\mathrm{III}}(r; H, c_{\mathrm{III}})$, and $\bar{f}'_{\mathrm{II}}(r; H, c_{\mathrm{II}}) = 1/h(r) + f'_{\mathrm{II}}(r; H, c_{\mathrm{II}})$ near $r = r_-$. The C^1 smoothness condition requires

$$\lim_{r\to(r_-)^+} \bar{f}'_{\mathrm{II}} = \lim_{r\to(r_-)^-} \bar{f}'_{\mathrm{III}} \Rightarrow \left(-Hr_- + \frac{c_{\mathrm{II}}}{r_-^2}\right)^2 = \left(Hr_- - \frac{c_{\mathrm{III}}}{r_-^2}\right)^2 \Rightarrow c_{\mathrm{III}} = c_{\mathrm{II}}.$$

The other solution $c_{\mathrm{III}} = 2r_-^3 H - c_{\mathrm{II}}$ is not satisfied because $c_{\mathrm{III}} > r_-^3 H$. We use $\lim_{r\to(r_-)^+} \tan(\bar{V}) = \lim_{r\to(r_-)^-} \tan(\bar{V})$ to determine $\bar{c}_{\mathrm{III}}$, and it gives

$$\exp\left(\bar{\alpha}\left(\int_{r_{\mathrm{II}}}^{r_-} \bar{f}'_{\mathrm{II}}(r)\,dr + r_{\mathrm{II}} + \frac{r_+^2}{r_+ - r_-} \ln|r_{\mathrm{II}} - r_+| - \frac{r_-^2}{r_+ - r_-} \ln|r_{\mathrm{II}} - r_-| + \bar{c}_{\mathrm{II}}\right)\right)$$
$$= \exp\left(\bar{\alpha}\left(\int_{r_{\mathrm{III}}}^{r_-} \bar{f}'_{\mathrm{III}}(r)\,dr + r_{\mathrm{III}} + \frac{r_+^2}{r_+ - r_-} \ln|r_{\mathrm{III}} - r_+| - \frac{r_-^2}{r_+ - r_-} \ln|r_{\mathrm{III}} - r_-| + \bar{c}_{\mathrm{III}}\right)\right).$$

When we take $c_{\mathrm{II}} = c_{\mathrm{III}}$, it implies that the expressions for $\bar{f}'_{\mathrm{II}}(r)$ and $\bar{f}'_{\mathrm{III}}(r)$ are the same, so we write them as $\bar{f}'(r)$. Hence

$$\bar{c}_{\mathrm{III}} = \bar{c}_{\mathrm{II}} + \int_{r_{\mathrm{II}}}^{r_{\mathrm{III}}} \bar{f}'(r)\,dr + (r_{\mathrm{II}} - r_{\mathrm{III}}) + \frac{r_+^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{II}} - r_+}{r_{\mathrm{III}} - r_+}\right| - \frac{r_-^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{II}} - r_-}{r_{\mathrm{III}} - r_-}\right|$$
$$= \bar{c}_{\mathrm{I}} + \int_{r_{\mathrm{I}}}^{r_{\mathrm{III}}} \bar{f}'(r)\,dr + (r_{\mathrm{I}} - r_{\mathrm{III}}) + \frac{r_+^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{I}} - r_+}{r_{\mathrm{III}} - r_+}\right| - \frac{r_-^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{I}} - r_-}{r_{\mathrm{III}} - r_-}\right|.$$

(B) <u>If $c_{\mathrm{I}} = C_H$</u>, then the SSCMC hypersurface in region II satisfies $\lim_{r\to R_H^+} f_{\mathrm{II}}(r) = -\infty$, so $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ tends to i^+ in the Penrose diagram, not passing through other regions.

(C) <u>If $r_+^3 H < c_{\mathrm{I}} < C_H$</u>, then the other end of the SSCMC hypersurface $\Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ in region II satisfies $\lim_{r\to(r_+)^-} f_{\mathrm{II}}(r) = -\infty$, and we will try to take some SSCMC hypersurface $\Sigma_{H,c_{\mathrm{I}'},\bar{c}_{\mathrm{I}'}}$ in region I' to glue them.

Recall the proof of case (C) in Proposition 3.18 and the formulae (4), (5), if we take the initial number $r_{\text{ini}} = r''$, then $\Sigma_{H,c_{\text{II}},\bar{c}_{\text{II}}}$ is the union of two graphs of functions $f(r; H, c_{\text{II}}, \bar{c}_{\text{II}})$ with the same c_{II} and $\bar{c}_{\text{II}}$ but with opposite sign of $f'(r; H, c_{\text{II}})$.

We write the SSCMC solution for $f'(r) < 0$ part near $r = r_+$ as $f_{\text{I}'}(r; H, c_{\text{I}'}, \bar{c}_{\text{I}'})$, $\bar{f}'_{\text{I}'}(r; H, c_{\text{I}'}) = 1/h(r) - f'_{\text{I}'}(r; H, c_{\text{I}'})$, and $\bar{f}'_{\text{II}}(r; H, c_{\text{II}}) = 1/h(r) - f'_{\text{II}}(r; H, c_{\text{II}})$. The C^1 smoothness condition requires

$$\lim_{r\to(r_+)^-} \bar{f}'_{\text{II}} = \lim_{r\to(r_+)^-} \bar{f}'_{\text{I}'} \Rightarrow \left(-Hr_+ + \frac{c_{\text{II}}}{r_+^2}\right)^2 = \left(-Hr_+ + \frac{c_{\text{I}'}}{r_+^2}\right)^2 \Rightarrow c_{\text{I}'} = c_{\text{II}}.$$

The other solution $c_{\text{I}'} = 2r_+^3 H - c_{\text{II}}$ is not satisfied because we require $c_{\text{I}'} > r_+^3 H$ by Proposition 3.7.

We use the condition $\lim_{r\to(r_+)^+} \tan(U) = \lim_{r\to(r_+)^-} \tan(U)$ to determine $c_{\text{I}'}$. Since

$$\begin{aligned}
&-\exp\left(-\alpha\left(-\int_{r''}^{r_+} \bar{f}'_{\text{II}}(r)\,dr - r'' - \frac{r_+^2}{r_+-r_-}\ln|r''-r_+| + \frac{r_-^2}{r_+-r_-}\ln|r''-r_-| + \bar{c}_{\text{II}}\right)\right)\\
&= -\exp\left(-\alpha\left(-\int_{r_{\text{I}'}}^{r_+} \bar{f}'_{\text{I}'}(r)\,dr - r_{\text{I}'} - \frac{r_+^2}{r_+-r_-}\ln|r_{\text{I}'}-r_+| + \frac{r_-^2}{r_+-r_-}\ln|r_{\text{I}'}-r_-| + \bar{c}_{\text{I}'}\right)\right),
\end{aligned}$$

and the expressions for $\bar{f}'_{\text{II}}(r)$ and $\bar{f}'_{\text{I}'}(r)$ are the same, so we write them as $\bar{f}'(r)$, and we have

$$\begin{aligned}
\bar{c}_{\text{I}'} &= \bar{c}_{\text{II}} + \int_{r_{\text{I}'}}^{r''} \bar{f}'(r)\,dr + (r_{\text{I}'} - r'') + \frac{r_+^2}{r_+-r_-}\ln\left|\frac{r_{\text{I}'}-r_+}{r''-r_+}\right| - \frac{r_-^2}{r_+-r_-}\ln\left|\frac{r_{\text{I}'}-r_-}{r''-r_-}\right|\\
&= \bar{c}_{\text{I}} + \int_{r_{\text{I}}}^{r''} \bar{f}'(r)\,dr + \int_{r_{\text{I}'}}^{r''} \bar{f}'(r)\,dr + (r_{\text{I}} + r_{\text{I}'} - 2r'')\\
&\quad + \frac{r_+^2}{r_+-r_-}\ln\left|\frac{(r_{\text{I}}-r_+)(r_{\text{I}'}-r_+)}{(r''-r_+)(r''-r_+)}\right| - \frac{r_-^2}{r_+-r_-}\ln\left|\frac{(r_{\text{I}}-r_-)(r_{\text{I}'}-r_-)}{(r''-r_-)(r''-r_-)}\right|.
\end{aligned}$$

Once we know that the SSCMC hypersurface is C^1 smooth, from the SSCMC equation (2), we get the union of SSCMC hypersurfaces is C^2, and the standard PDE theory (see [4, Theorem 6.17.] for example) implies that the SSCMC hypersurface is C^∞. □

THEOREM 4.2. *Given constant mean curvature $H \in \mathbb{R}$, $c_{\text{I}} = r_+^3 H$, and $\bar{c}_{\text{I}} \in \mathbb{R}$, it will determine an SSCMC hypersurface $\Sigma_{H,c_{\text{I}},\bar{c}_{\text{I}}}$ in region* I. *We can take an SSCMC hypersurface $\Sigma_{H,c_{\text{I}'},\bar{c}_{\text{I}'}}$ in region* I' *with $c_{\text{I}'} = c_{\text{I}}$ and with some $\bar{c}_{\text{I}'}$ determined by $\bar{c}_{\text{I}}$ such that $\Sigma_{H,c_{\text{I}},\bar{c}_{\text{I}}} \cup \Sigma_{H,c_{\text{I}'},\bar{c}_{\text{I}'}}$ is a smooth SSCMC hypersurface in the extended RN spacetime.*

Proof: When $c_{\text{I}} = r_+^3 H$, the orders of $f'_{\text{I}}(r)$ and $f'_{\text{I}'}(r)$ near $r = r_+$ are $O(|r - r_+|^{-\frac{1}{2}})$ such that $\lim_{r\to(r_+)^+} f_{\text{I}}(r)$ and $\lim_{r\to(r_+)^+} f_{\text{I}'}(r)$ are finite. In region I, we

observe

$$\tan(U) = \sqrt{r - r_+}\exp\left(-\alpha\left(\int_{r_{\mathrm{I}}}^{r_+} f_{\mathrm{I}}'(r)\,dr + \bar{c}_{\mathrm{I}} - r_+ + \frac{r_-^2}{r_+ - r_-}\ln|r - r_+|\right)\right),$$

$$\tan(V) = \sqrt{r - r_+}\exp\left(\alpha\left(\int_{r_{\mathrm{I}}}^{r_+} f_{\mathrm{I}}'(r)\,dr + \bar{c}_{\mathrm{I}} + r_+ - \frac{r_-^2}{r_+ - r_-}\ln|r - r_+|\right)\right),$$

and

$$\left.\frac{dV}{dU}\right|_{r=(r_+)^+} = \left.\frac{\frac{dV}{dr}}{\frac{dU}{dr}}\right|_{r=(r_+)^+} = \exp\left(2\alpha\int_{r_{\mathrm{I}}}^{r_+} f_{\mathrm{I}}'(r)\,dr + \bar{c}_{\mathrm{I}}\right).$$

In region I', we have

$$\tan(U) = -\sqrt{r - r_+}\exp\left(-\alpha\left(\int_{r_{\mathrm{I}'}}^{r_+} f_{\mathrm{I}'}'(r)\,dr + \bar{c}_{\mathrm{I}'} - r_+ + \frac{r_-^2}{r_+ - r_-}\ln|r - r_+|\right)\right),$$

$$\tan(V) = -\sqrt{r - r_+}\exp\left(\alpha\left(\int_{r_{\mathrm{I}'}}^{r_+} f_{\mathrm{I}'}'(r)\,dr + \bar{c}_{\mathrm{I}'} + r_+ - \frac{r_-^2}{r_+ - r_-}\ln|r - r_+|\right)\right),$$

and

$$\left.\frac{dV}{dU}\right|_{r=(r_+)^-} = \left.\frac{\frac{dV}{dr}}{\frac{dU}{dr}}\right|_{r=(r_+)^-} = \exp\left(2\alpha\int_{r_{\mathrm{I}'}}^{r_+} f_{\mathrm{I}'}'(r)\,dr + \bar{c}_{\mathrm{I}'}\right).$$

Since $f_{\mathrm{I}}'(r)$ and $f_{\mathrm{I}'}'(r)$ have the same expressions, we write them as $f'(r)$. We can choose

$$\bar{c}_{\mathrm{I}'} = \bar{c}_{\mathrm{I}} + 2\alpha\int_{r_{\mathrm{I}}}^{r_{\mathrm{I}'}} f'(r)\,dr$$

such that $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{I}'},\bar{c}_{\mathrm{I}'}}$ is a C^1 and thus C^∞ SSCMC hypersurface. □

THEOREM 4.3. *Given constant mean curvature $H \in \mathbb{R}$, $c_{\mathrm{I}} < r_+^3 H$, and $\bar{c}_{\mathrm{I}} \in \mathbb{R}$, it will determine an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}}$ in region* I. *We can take an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{II}'},\bar{c}_{\mathrm{II}'}}$ in region* II' *with $c_{\mathrm{II}'} = c_{\mathrm{I}}$ and with some $\bar{c}_{\mathrm{II}'}$ determined by $\bar{c}_{\mathrm{I}}$ such that $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}'},\bar{c}_{\mathrm{II}'}}$ is a smooth SSCMC hypesurface. Furthermore, suppose that $c_H = \min_{r\in(r_-,r_+)} Hr^3 - r(-r^2 + 2mr - e^2)^{\frac{1}{2}}$.*

(A) *If $c_H < c_{\mathrm{I}} < r_+^3 H$, then we can take an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{I}'},\bar{c}_{\mathrm{I}'}}$ in region* I' *with $c_{\mathrm{I}'} = c_{\mathrm{I}}$ and with some $\bar{c}_{\mathrm{I}'}$ determined by $\bar{c}_{\mathrm{I}}$ such that $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}'},\bar{c}_{\mathrm{II}'}} \cup \Sigma_{H,c_{\mathrm{I}'},\bar{c}_{\mathrm{I}'}}$ is a smooth SSCMC hypersurface in the extended RN spacetime.*

(B) *If $c_{\mathrm{I}} = c_H$, then $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}'},\bar{c}_{\mathrm{II}'}}$ is a smooth SSCMC hypersurface ranging from* I *to* II' *in the extended RN spacetime.*

(C) *If $c_{\mathrm{I}} < c_H$, then we can take an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{III}},\bar{c}_{\mathrm{III}}}$ in region* III *with $c_{\mathrm{III}} = c_{\mathrm{I}}$ and with some $\bar{c}_{\mathrm{III}}$ determined by $\bar{c}_{\mathrm{I}}$ such that $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}'},\bar{c}_{\mathrm{II}'}} \cup \Sigma_{H,c_{\mathrm{III}},\bar{c}_{\mathrm{III}}}$ is a smooth SSCMC hypersurface in the extended RN spacetime.*

The proof of this theorem is similar to the proof in Theorem 4.1. By examining the Taylor expansions of SSCMC solutions $f(r; H, c, \bar{c})$ near coordinate singularities, we can get the C^1 smoothness conditions, and by PDE theory, we prove the C^∞ smoothness of the extended SSCMC hypersurfaces. Here we summarize the explicit relations on $\bar{c}$:

(A) If $c_{\mathrm{I}} < r_+^3 H$, then

$$\bar{c}_{\mathrm{II}'} = \bar{c}_{\mathrm{I}} + \int_{r_{\mathrm{I}}}^{r_{\mathrm{II}'}} \bar{f}'(r)\,dr - (r_{\mathrm{I}} - r_{\mathrm{II}'}) - \frac{r_+^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{I}} - r_+}{r_{\mathrm{II}'} - r_+}\right| + \frac{r_-^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{I}} - r_-}{r_{\mathrm{II}'} - r_-}\right|$$

and (take $r_{\mathrm{II}'} = r''$)

$$\bar{c}_{\mathrm{I}'} = \bar{c}_{\mathrm{I}} + \int_{r_{\mathrm{I}}}^{r''} \bar{f}'(r)\,dr + \int_{r_{\mathrm{I}'}}^{r''} \bar{f}'(r)\,dr + (2r'' - r_{\mathrm{I}} - r_{\mathrm{I}'}) - \frac{r_+^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{I}} - r_+}{r'' - r_+} \cdot \frac{r_{\mathrm{I}'} - r_+}{r'' - r_+}\right| + \frac{r_-^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{I}} - r_-}{r'' - r_-} \cdot \frac{r_{\mathrm{I}'} - r_-}{r'' - r_-}\right|.$$

(C) If $c_{\mathrm{I}} < c_H$, then

$$\bar{c}_{\mathrm{III}} = \bar{c}_{\mathrm{I}} + \int_{r_{\mathrm{I}}}^{r_{\mathrm{III}}} \bar{f}'(r)\,dr - (r_{\mathrm{I}} - r_{\mathrm{III}}) - \frac{r_+^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{I}} - r_+}{r_{\mathrm{III}} - r_+}\right| + \frac{r_-^2}{r_+ - r_-} \ln\left|\frac{r_{\mathrm{I}} - r_-}{r_{\mathrm{III}} - r_-}\right|.$$

Fig. 7 illustrates each case of SSCMC hypersurfaces according to the above discussion.

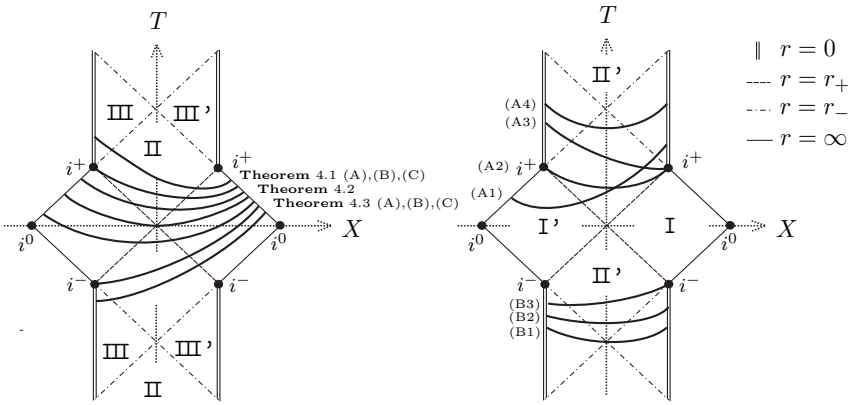


Fig. 7. Left: Given $\Sigma_{H,c,\bar{c}}$ in region I, we can construct the extended smooth SSCMC hypersurface in the extended RN spacetime according to c and $\bar{c}$. Right: Given $\Sigma_{H,c,\bar{c}}$ in different regions, we can also construct the extended smooth SSCMC hypersurface.

From Theorems 4.1– 4.3, we construct many extended smooth SSCMC hypersurfaces in the extended RN spacetime. If we start from SSCMC hypersurfaces in different regions, what are new extended smooth SSCMC hypersurfaces? Here we summarize the results:

(A) We start from an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ in region II.

(A1) If $c_{\mathrm{II}} > C_H$ and $f'(r) < 0$ in region II, we can extend $\Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ to regions I' and III' and get the extended SSCMC hypersurface $\Sigma_{H,c_{\mathrm{I}'},\bar{c}_{\mathrm{I}'}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}} \cup \Sigma_{H,c_{\mathrm{III}'},\bar{c}_{\mathrm{III}'}}$.

(A2) If $c_{\mathrm{II}} = C_H$ and $|f'(r)| = \infty$, then $\Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ is in fact the cylindrical hypersurface $(t, r = R_H, \theta, \phi)$.

(A3) If $c_{\mathrm{II}} = C_H$ and $|f'(r)| \neq \infty$ for all r, besides $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$ is stated in Theorem 4.1 (B), we can get different types of SSCMC hypersurface such as $\Sigma_{H,c_{\mathrm{I}'},\bar{c}_{\mathrm{I}'}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}}$, $\Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}} \cup \Sigma_{H,c_{\mathrm{III}},\bar{c}_{\mathrm{III}}}$, and $\Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}} \cup \Sigma_{H,c_{\mathrm{III}'},\bar{c}_{\mathrm{III}'}}$.

(A4) If $r_-^3 H < c_{\mathrm{II}} < C_H$ and $|f'(r)| \neq \infty$ for all r, besides $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}} \cup \Sigma_{H,c_{\mathrm{I}'},\bar{c}_{\mathrm{I}'}}$ is stated in Theorem 4.1 (C), another type of SSCMC hypersurface is $\Sigma_{H,c_{\mathrm{III}},\bar{c}_{\mathrm{III}}} \cup \Sigma_{H,c_{\mathrm{II}},\bar{c}_{\mathrm{II}}} \cup \Sigma_{H,c_{\mathrm{III}'},\bar{c}_{\mathrm{III}'}}$.

(B) We start from an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{III}},\bar{c}_{\mathrm{III}}}$ in region III.

(B1) If $c_{\mathrm{III}} = r_-^3 H$, we get $\Sigma_{H,c_{\mathrm{III}},\bar{c}_{\mathrm{III}}} \cup \Sigma_{H,c_{\mathrm{III}'},\bar{c}_{\mathrm{III}'}}$.

(B2) If $c_H < c_{\mathrm{III}} < r_-^3 H$, we get $\Sigma_{H,c_{\mathrm{III}},\bar{c}_{\mathrm{III}}} \cup \Sigma_{H,c_{\mathrm{II}'},\bar{c}_{\mathrm{II}'}} \cup \Sigma_{H,c_{\mathrm{III}'},\bar{c}_{\mathrm{III}'}}$.

(B3) If $c_{\mathrm{III}} = c_H$, we get $\Sigma_{H,c_{\mathrm{III}},\bar{c}_{\mathrm{III}}} \cup \Sigma_{H,c_{\mathrm{II}'},\bar{c}_{\mathrm{II}'}}$.

SSCMC hypersurfaces starting from regions I', II', or III' can be discussed by a similar procedure as the one used in Theorems 4.1–4.3. As a remark, here we provide another viewpoint to construct the SSCMC hypersurface starting from these regions. For example, we start an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{I}'},\bar{c}_{\mathrm{I}'}}$ in region I' with coordinate $(T = F(X), X, \theta, \phi)$, where $X \leq 0$. We use the T-axis symmetry to get an SSCMC hypersurface $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}}$ in region I. In other words, $\Sigma_{H,c_{\mathrm{I}},\bar{c}_{\mathrm{I}}}$ is of coordinates $(T = F_1(X) \overset{\text{def.}}{=\!=} F(-X), X, \theta, \phi)$, where $X \geq 0$, and furthermore, from the formula of SSCMC solutions, we know that $c_{\mathrm{I}} = c_{\mathrm{I}'}$ and $\bar{c}_{\mathrm{I}} = \bar{c}_{\mathrm{I}'}$. From Theorems 4.1–4.3, we get the extended smooth SSCMC hypersurface determined by c_{I} and $\bar{c}_{\mathrm{I}'}$ with coordinates $(T = F_2(X), X, \theta, \phi)$, where $F_2(X)$ is the extended smooth function containing $F_1(X)$. Finally, we use the T-axis symmetry again to get $\Sigma : (T = \bar{F}(X) \overset{\text{def.}}{=\!=} F_2(-X), X, \theta, \phi)$, which is an extended smooth SSCMC hypersurface containing $\Sigma_{H,c_{\mathrm{I}'},\bar{c}_{\mathrm{I}'}}$ in the extended RN spacetime.

5. Initial value problem for SSCMC equation in the extended RN spacetime

In this section, we will first formulate the SSCMC initial value problem as follows.

SS-CMC INITIAL VALUE PROBLEM. *Given $H \in \mathbb{R}$, a point (T_0, X_0), and a value V_0 with $1 - V_0^2 > 0$ in the Penrose diagram of the extended RN spacetime with*

the charge smaller than the mass, does there exist a unique function $T = T(X)$ *satisfying the SSCMC equation* (2), $T(X_0) = T_0$, $T'(X_0) = V_0$, *and* $1 - (T'(X))^2 > 0$ *for all* $T(X)$ *is defined?*

If the SSCMC initial value problem is true, then $\Sigma : (T = T(X), X, \theta, \phi)$ is an SSCMC hypersurface with constant mean curvature H in the extended RN spacetime. From the discussion in Section 4, we are ready to answer this SSCMC initial value problem.

MAIN THEOREM. *The initial value problem for the spacelike, spherically symmetric, constant mean curvature hypersurface equation in the maximally extended RN spacetime with the charge smaller than the mass is solvable and the solution is unique.*

Proof: Suppose that (T_0, X_0) is located at (t_0, r_0) for some RN spacetime in the standard coordinates region. The initial value problem in the extended RN spacetime can be equivalently changed as the initial value problem in the standard coordinates with condition (t_0, r_0) and $f'(r_0) = v_0$. Remark that v_0 may be ∞ or $-\infty$ if the initial point is in regions II or II'. According to the discussion in Section 4, there exists a unique SSCMC solution $f(r; H, c, \bar{c})$ satisfying $f(r_0) = t_0$ and $f'(r_0) = v_0$, and the graph of the function $f(r; H, c, \bar{c})$ can be uniquely smoothly extended to other regions in the maximally extended RN spacetime. Therefore, we have proved the main theorem. □

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