Chapter 2 Regular Surfaces

2.1 Introduction

In this chapter, we will introduce the concept of a regular surface in $\mathbb{R}^3$, and then discuss differentiable functions, the first fundamental form, and observe the orientation property on regular surfaces.

2.2 Regular Surfaces; Inverse Images of Regular Values

Roughly speaking, we want to define geometric objects obtained by taking pieces of a plane, deforming and arranging them so that the figure has no sharp points, edges, or self-intersections. We can do calculus on such objects called regular surfaces.

Definition 1 (page 52)**.** A subset $S \subset \mathbb{R}^3$ is a *regular surface* (正則曲面) if for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

(a) The map $\mathbf{x}$ is smooth. If we write

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U,$$

the functions $x(u, v), y(u, v)$, and $z(u, v)$ have continuous partial derivatives of all orders in U.

(b) The map $\mathbf{x}$ is homeomorphism (同胚). That is, the map $\mathbf{x}$ has an inverse map $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous.

(c) For each $q \in U$, the differential map (微分映射) $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

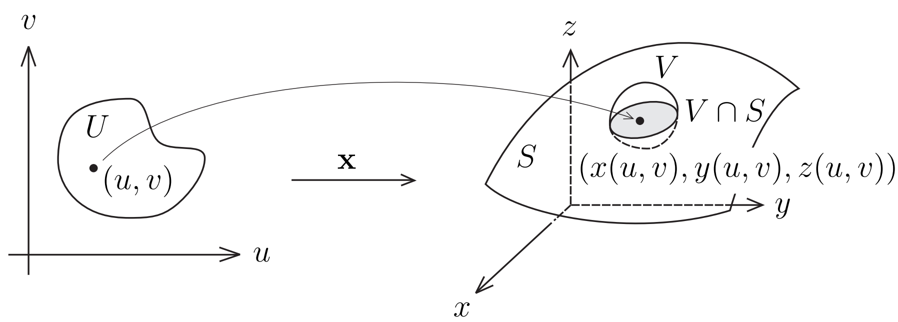

Figure 1: A regular surface.

The mapping $\mathbf{x}$ is called a *parametrization* (參數化) or a *system of local coordinates* (局部坐標系) in a neighborhood of p. The neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood* (坐標鄰域).

We will discuss the condition (a) and (c) in this section, and leave the discussion on the condition (b) in next section. Let us discuss the condition (c) first. We choose the canonical bases $\{\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)\}$ of $\mathbb{R}^2$ with coordinates (u,v), and $\{\mathbf{f}_1 = (1,0,0), \mathbf{f}_2 = (0,1,0), \mathbf{f}_3 = (0,0,1)\}$ of $\mathbb{R}^3$ with coordinates (x,y,z). Let $q = (u_0, v_0)$, then the space curve $\boldsymbol{\alpha}(u) = \mathbf{x}(u, v_0) = (x(u,v_0), y(u,v_0), z(u,v_0))$ has image lies on S and has at $\mathbf{x}(q)$ the tangent vector

$$\mathbf{x}_u\big|_{q=(u_0,v_0)} = \frac{\partial \mathbf{x}}{\partial u}\bigg|_{q=(u_0,v_0)} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)\bigg|_{q=(u_0,v_0)} = \mathrm{d}\mathbf{x}_q(\mathbf{e}_1).$$

Similarly, the space curve $\boldsymbol{\beta}(v) = \mathbf{x}(u_0, v) = (x(u_0,v), y(u_0,v), z(u_0,v))$ has image lies on S and has at $\mathbf{x}(q)$ the tangent vector

$$\mathbf{x}_v\big|_{q=(u_0,v_0)} = \frac{\partial \mathbf{x}}{\partial v}\bigg|_{q=(u_0,v_0)} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\bigg|_{q=(u_0,v_0)} = \mathrm{d}\mathbf{x}_q(\mathbf{e}_2).$$

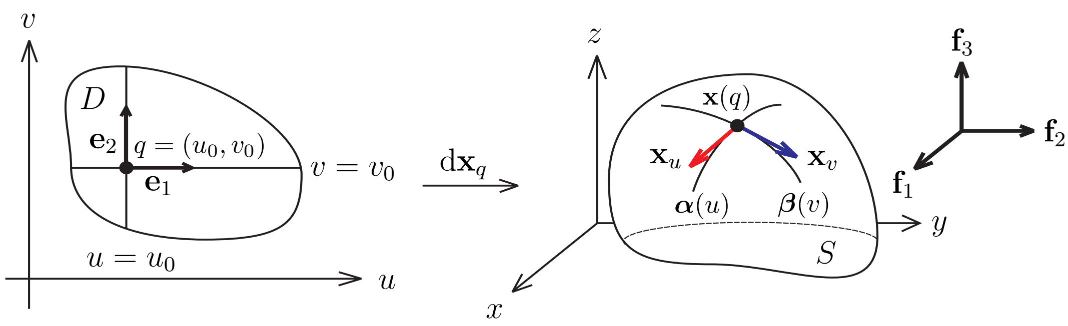

Figure 2: The differential map $\mathrm{d}\mathbf{x}_q$.

Thus, the differential map $\mathrm{d}\mathbf{x}_q$, can be written as a matrix form

$$[\mathrm{d}\mathbf{x}_q]_{3\times 2} = [\,\mathrm{d}\mathbf{x}_q(\mathbf{e}_1)\,|\,\mathbf{x}_q(\mathbf{e}_2)\,] = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}.$$

The differential map $\mathrm{d}\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one means that two column vectors of the matrix $[\mathrm{d}\mathbf{x}_q]$ are *linearly independent* (線性獨立). It is also equivalent to $\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \neq \mathbf{0}$, or one of the minors of order 2 of the matrix $\mathrm{d}\mathbf{x}_q$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad \frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}, \qquad \frac{\partial(x,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix},$$

be different from zero at q.

What surfaces are regular surfaces? In the following paragraphs, we will prove two results. One result is that the graph of a smooth function is a regular surface. The other result is that the surface determined by a regular value of an implicit function is a regular surface.

Proposition 2 (page 58)**.** *If $f : U \to \mathbb{R}$ is a smooth function in an open set U of $\mathbb{R}^2$, then the graph of f, the subset of $\mathbb{R}^3$ given by $(x, y, f(x, y))$ for $(x, y) \in U$, is a regular surface.*

Proof. It suffices to show that the map $\mathbf{x} : U \to \mathbb{R}^3$ given by

$$\mathbf{x}(x, y) = (x, y, f(x, y))$$

satisfies all conditions of the regular surface.

(a) Since $f(x, y)$ is smooth, the map $\mathbf{x}(x, y)$ is smooth as well.

(b) If $(x_1, y_1) \neq (x_2, y_2)$, then $(x_1, y_1, f(x_1, y_1)) \neq (x_2, y_2, f(x_2, y_2))$. So the map $\mathbf{x}$ is one-to-one, and $\mathbf{x}^{-1}$ is well-defined. Since $\mathbf{x}^{-1}$ is the restriction to the graph of f of the continuous projection of $\mathbb{R}^3$ onto the xy plane, $\mathbf{x}^{-1}$ is continuous.

(c) Since $\frac{\partial \mathbf{x}}{\partial x} \wedge \frac{\partial \mathbf{x}}{\partial y} = (1, 0, f_x) \wedge (0, 1, f_y) = (-f_x, -f_y, 1) \neq \mathbf{0}$, we know that the differential $\mathrm{d}\mathbf{x}_q$ is one-to-one.

□

☐ 光滑函數的圖形必爲正則曲面。

Example 3. The *Elliptic paraboloid* (橢圓拋物面) $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ and *hyperbolic paraboloid* (雙曲拋物面) $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ are regular surfaces because they are graphs of functions $f(x, y) = c(\frac{x^2}{a^2} + \frac{y^2}{b^2})$ and $f(x, y) = c(\frac{x^2}{a^2} - \frac{y^2}{b^2})$, respectively.

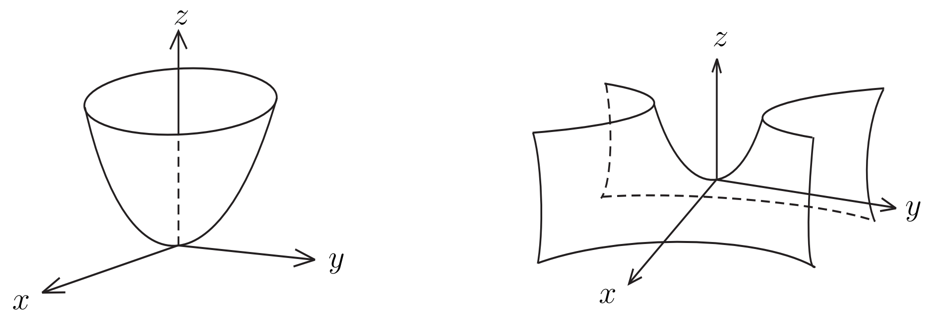

Figure 3: Elliptic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$; Hyperbolic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}, c < 0$.

Definition 4 (page 58)**.** Given a smooth map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set U of $\mathbb{R}^n$, we say that $p \in U$ is a *critical point* (臨界點) of F if the differential $\mathrm{d}F_p : \mathbb{R}^n \to \mathbb{R}^m$ is *not* a surjective (or onto) mapping. The image $F(p) \in \mathbb{R}^m$ of a critical point is called a *critical value* (臨界值) of F. A point of $\mathbb{R}^m$ which is *not* a critical value is called a *regular value* (正則值) of F.

The terminology is motivated by the particular case in which $F : U \subset \mathbb{R} \to \mathbb{R}$ is a real-valued function of a real variable. A point $x_0 \in U$ is critical if $F'(x_0) = 0$, that is, if the differential $\mathrm{d}F_{x_0}$ carries all the vectors in $\mathbb{R}$ to the zero vector.

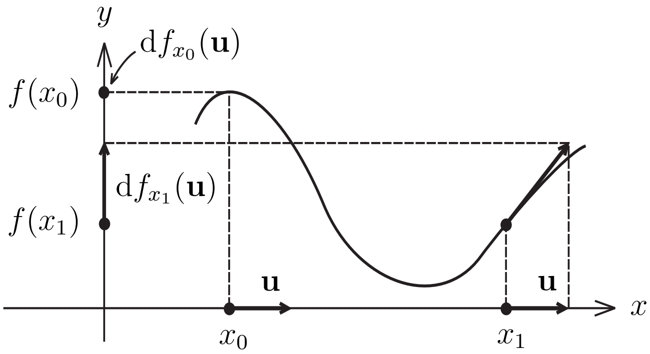

Figure 4: Critical point and regular point.

If $F : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a smooth function, then $\mathrm{d}F_p$ applied to the vector $\mathbf{e}_1 = (1, 0, 0)$ is obtained by calculating the tangent vector at $F(p)$ to the curve $x \to F(x, y_0, z_0)$. It follows that

$$\mathrm{d}F_p(\mathbf{e}_1) = \frac{\partial F}{\partial x}(x_0, y_0, z_0) = F_x(p).$$

Similarly, for $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$, we have

$$\mathrm{d}F_p(\mathbf{e}_2) = \frac{\partial F}{\partial y}(x_0, y_0, z_0) = F_y(p),$$
$$\mathrm{d}F_p(\mathbf{e}_3) = \frac{\partial F}{\partial z}(x_0, y_0, z_0) = F_z(p).$$

We conclude that the matrix $[\mathrm{d}F_p]$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is given by

$$[\mathrm{d}F_p]_{1\times 3} = [\,\mathrm{d}F_p(\mathbf{e}_1)\,|\,\mathrm{d}F_p(\mathbf{e}_2)\,|\,\mathrm{d}F_p(\mathbf{e}_3)\,] = [\,F_x(p)\ F_y(p)\ F_z(p)\,].$$

Notice that $\mathrm{d}F_p$ is not surjective is equivalent that $F_x(p) = F_y(p) = F_z(p) = 0$. Hence $C \in F(U)$ is a regular value of $F : U \subset \mathbb{R}^3 \to \mathbb{R}$ if and only if F_x, F_y, and F_z do *not* vanish simultaneously at *any* point in the inverse image

$$F^{-1}(C) = \{(x, y, z) \in U : F(x, y, z) = C\}.$$

Proposition 5 (page 59)**.** *If $F : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and $C \in F(U)$ is a regular value of F, then $F^{-1}(C)$ is a regular surface in $\mathbb{R}^3$.*

Proof. Let $p = (x_0, y_0, z_0)$ be a point of $F^{-1}(C)$. Since C is a regular value of f, without loss of generality, we assume that $F_z \neq 0$ at p. Consider the following mapping:

$$\begin{aligned} \overline{F} : U \subset \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (u, v, t) = (x, y, F(x, y, z)), \end{aligned}$$

Then the differential of $\overline{F}$ at p is given by

$$\mathrm{d}\overline{F}_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ F_x & F_y & F_z \end{pmatrix}.$$

We know that $\det(\mathrm{d}F_p) = F_z \neq 0$.

By the *Implicit Function Theorem* (隱函數定理), there exists neighborhoods V of p and W of $\overline{F}(p)$ such that $\overline{F} : V \to W$ is invertible and the inverse $\overline{F}^{-1} : W \to V$ is differentiable. It follows that the coordinate function of $\overline{F}^{-1}$, that is, the functions

$$x = u, \quad y = v, \quad z = g(u, v, t), \quad (u, v, t) \in W,$$

are differentiable. In particular, $z = g(u, v, C) = h(x, y)$ is a differentiable function defined in the projection of V onto the xy plane. Since

$$\overline{F}(F^{-1}(C) \cap V) = W \cap \{(u, v, t) | t = C\},$$

we conclude that the graph of h is $F^{-1}(C) \cap V$. By Proposition 2, $F^{-1}(C) \cap V$ is a coordinate neighborhood of p. Therefore, every $p \in F^{-1}(C)$ can be covered by a coordinate neighborhood, and so $F^{-1}(C)$ is a regular surface. $\square$

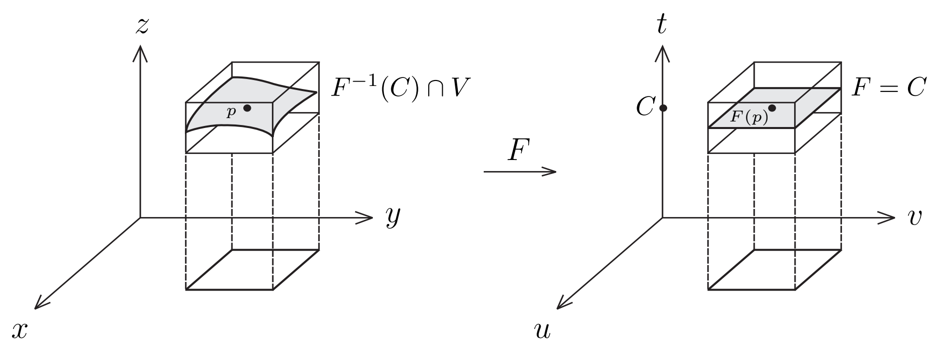

Figure 5: A regular value of a smooth function F is a regular surface.

Example 6 (page 61)**.** The *ellipsoid* (橢球) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is a regular surface.

Solution. Consider the smooth function $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$. We compute $(F_x,F_y,F_z)=(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2})$. Since $(F_x,F_y,F_z)=(0,0,0)$ if and only if $(x,y,z)=(0,0,0)$, we know that $F(0,0,0)=0$ is the only critical value, and $F(x,y,z)=C, C>0$ is a regular value of F. Taking $C=1$ and we get the ellipsoid is a regular surface.

In particular, the sphere $x^2+y^2+z^2=1$ is a regular surface since $a=b=c=1$.

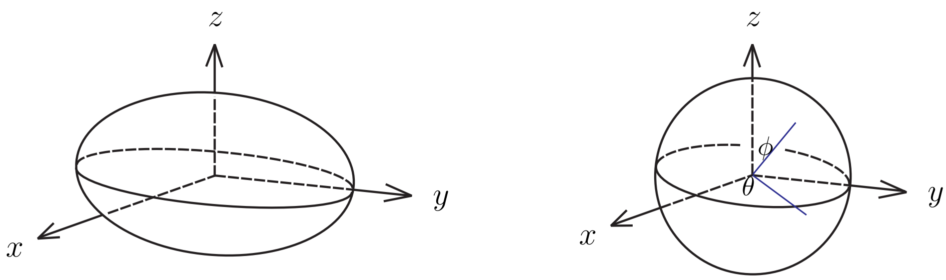

Figure 6: Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and sphere $x^2+y^2+z^2=1$.

For most applications, it is convenient to relate parameterizations to the *spherical coordinates* (球極坐標) on $\mathbb{S}^2$. Let $V=\{(\phi,\theta);0<\phi<\pi,0<\theta<2\pi\}$ and let $\mathbf{x}:V\to\mathbb{R}^2$ be given by

$$\mathbf{x}(\phi,\theta)=(\sin\phi\cos\theta,\sin\phi\sin\theta,\cos\phi).$$

The parameter ϕ is called the *colatitude* (餘緯) (the complement of the latitude) and θ the *longitude* (經度).

☐ 球極坐標的 ϕ 與 θ 的設定並沒有統一的規定, 應對照前後文理解。

Exercise. Show that the *hyperboloid of one sheet* (單葉雙曲面) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is a regular surface.

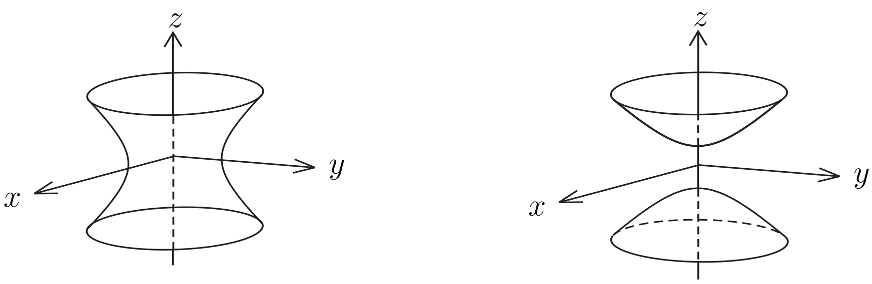

Figure 7: Hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ and hyperboloid of two sheets $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Exercise (page 61). The *hyperboloid of two sheets* (雙葉雙曲面) $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is a regular surface. The surface is not *path-connected* (路徑連通). That is, given two points in two distinct sheets ($z \geq 1$ part and $z \leq -1$ part), it is impossible to join them by a continuous curve $\boldsymbol{\alpha}(t) = (x(t), y(t), z(t))$ contained in the surface.

Example 7 (page 61). The *torus* (輪胎面) T is generated by rotating a circle $\mathbb{S}^1$ of radius r about a straight line belonging to the plane of the circle and at a distance $a > r$ away from the center of the circle.

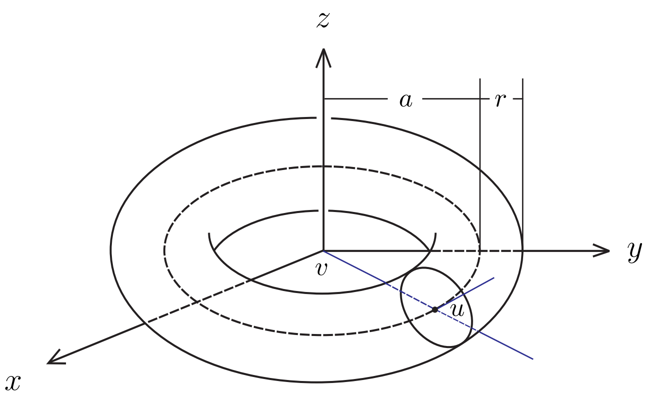

Figure 8: Torus.

Let $\mathbb{S}^1$ be the circle in the yz plane with its center in the point $(0, a, 0)$, then $\mathbb{S}^1$ is given by $(y-a)^2 + z^2 = r^2$, and the points of the torus T obtained by rotating this circle about the z axis satisfy the equation

$$\left(\sqrt{x^2+y^2} - a\right)^2 + z^2 = r^2.$$

To show that the torus T is a regular surface, consider the function $F(x, y, z) = \left(\sqrt{x^2+y^2} - a\right)^2 + z^2$, which is differentiable for $(x, y) \neq (0, 0)$. We compute

$$\frac{\partial F}{\partial x} = \frac{2x(\sqrt{x^2+y^2} - a)}{\sqrt{x^2+y^2}}, \quad \frac{\partial F}{\partial y} = \frac{2y(\sqrt{x^2+y^2} - a)}{\sqrt{x^2+y^2}}, \quad \text{and} \quad \frac{\partial F}{\partial z} = 2z.$$

The critical points of F will be $(0, 0, 0)$ and all points satisfy $x^2+y^2 = a^2, z = 0$. We know that 0 and a^2 are critical values of F, and r^2 is a regular value of F. Therefore, the torus T is a regular surface.

Remark that one parametrization for the torus T is given by

$$\mathbf{x}(u, v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u),$$

where $0 < u < 2\pi, 0 < v < 2\pi$.

Proposition 2 says that the graph of a smooth function is a regular surface. The following proposition provides a local version of this; that is, any regular surface is locally the graph of a smooth function.

Proposition 8 (page 63). *Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then there exists a neighborhood V of p in S such that V is the graph of a differentiable function which has one of the following three forms: $z = f(x, y), y = g(x, z), x = h(y, z)$.*

Proof. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization of S in p, and write $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in U$. By condition (c) of the definition of a regular surface, one of the Jacobian determinants

$$\frac{\partial(x, y)}{\partial(u, v)}, \quad \frac{\partial(y, z)}{\partial(u, v)}, \quad \frac{\partial(z, x)}{\partial(u, v)}$$

is *not* zero at $\mathbf{x}^{-1}(p) = q$.

Suppose first that $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$. Consider the map $\pi \circ \mathbf{x} : U \to \mathbb{R}^2$, where π is the projection $\pi(x, y, z) = (x, y)$, then $\pi \circ \mathbf{x}(u, v) = (x(u, v), y(u, v))$. By the inverse function theorem, there exist neighborhood V_1 of q and V_2 of $\pi \circ \mathbf{x}(q)$ such that $\pi \circ \mathbf{x}$ map V_1 diffeomorphically onto V_2. It follows that π restricted to $\mathbf{x}(V_1) = V$ is one-to-one and there is a smooth inverse map $(\pi \circ \mathbf{x})^{-1} : V_2 \to V_1$. Since $\mathbf{x}$ is a homeomorphism, V is a neighborhood of p in S. Now, if we compose the map $(\pi \circ \mathbf{x})^{-1} : (x, y) \to (u(x, y), v(x, y))$ with the function $(u, v) \to z(u, v)$, we find that V is the graph of the smooth function $z = z(u(x, y), v(x, y)) = f(x, y)$, and this settles the first case. The remaining case can be treated in the same way, yielding $x = h(y, z)$ and $y = g(x, z)$. $\square$

Example 9 (page 64). The one-sheet cone C, given by

$$z = \sqrt{x^2 + y^2}, \quad (x, y) \in \mathbb{R}^2,$$

is *not* a regular surface.

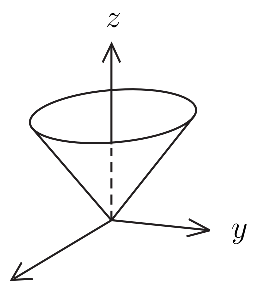

Figure 9: Cone.

Observe that we *cannot* conclude this from the fact alone that the natural parametrization

$$(x, y) \to \left(x, y, \sqrt{x^2 + y^2}\right)$$

is *not* differentiable. There could be other parameterizations satisfying all conditions in the definition of a regular surface.

To show that this is not the case, suppose that C is a regular surface, then in a neighborhood of $(0,0,0) \in C$, by Proposition 8, the graph of a differentiable function having one of three form $z = f(x,y), y = h(x,z), x = g(y,z)$. The last two forms can be discarded by the fact that the projections of C over xz and yz planes are *not* one-to-one. So the first form would have to agree, in a neighborhood of $(0,0,0)$, with $z = \sqrt{x^2+y^2}$. Since $z = \sqrt{x^2+y^2}$ is *not* differentiable at $(0,0)$, we know that C is not a regular surface.

Example 10 (Surfaces of Revolution, page 76)**.** Let $S \subset \mathbb{R}^3$ be the set obtained by rotating a regular smooth plane curve C about an axis in the plane which does *not* meet the curve. For example, we take the xz plane as the plane of the curve and the z axis as the rotation axis. Let

$$x = f(v), \quad z = g(v), \quad a < v < b, \quad f(v) > 0,$$

be a parametrization for C and denote by u the rotation angle about the z axis. Thus, we obtain a map

$$\mathbf{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

from the open set $U = \{(u,v) \in \mathbb{R}^2, 0 < u < 2\pi, a < v < b\}$ into S.

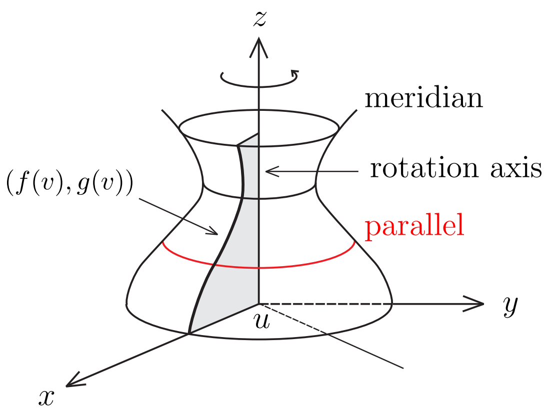

Figure 10: A surface of revolution.

We will show that $\mathbf{x}$ satisfies the conditions for a parametrization in the definition of a regular surface.

(a) Since both $f(v)$ and $g(v)$ are smooth functions, we know that $\mathbf{x}(u,v)$ is smooth.

(b) To show that $\mathbf{x}$ is a homeomorphism, we first show that $\mathbf{x}$ is one-to-one. In fact, since $(f(v), g(v))$ is a parametrization of C, given z and $x^2+y^2 = (f(v))^2$, we can determine v uniquely, and so is u. Thus, $\mathbf{x}$ is one-to-one.

Since $(f(v), g(v))$ is a parametrization of C, v is a continuous function of z and of $\sqrt{x^2+y^2}$ and thus a continuous function of (x, y, z).

To prove that $\mathbf{x}^{-1}$ is continuous, it remains to show that u is a continuous function of (x, y, z). To prove this, we first observe that if $u \neq \pi$, since $f(v) = 0$, we have

$$\tan\frac{u}{2} = \frac{\sin\frac{u}{2}}{\cos\frac{u}{2}} = \frac{2\sin\frac{u}{2}\cos\frac{u}{2}}{2\cos^2\frac{u}{2}} = \frac{\sin u}{1+\cos u} = \frac{\frac{y}{f(v)}}{1+\frac{x}{f(v)}} = \frac{y}{x+\sqrt{x^2+y^2}};$$

hence

$$u = 2\tan^{-1}\left(\frac{y}{x+\sqrt{x^2+y^2}}\right).$$

Thus, if $u \neq \pi$, u is continuous function of (x, y, z). Similarly, if u is in a small interval about π, we obtain

$$u = 2\cot^{-1}\left(\frac{y}{-x+\sqrt{x^2+y^2}}\right).$$

Thus, u is a continuous function of (x, y, z). This shows that $\mathbf{x}^{-1}$ is continuous.

(c) We compute

$$\begin{aligned}
\mathbf{x}_u(u,v) &= (-f(v)\sin u, f(v)\cos u, 0)\\
\mathbf{x}_v(u,v) &= (f'(v)\cos u, f'(v)\sin u, g'(v))\\
\mathbf{x}_u \wedge \mathbf{x}_v &= (f(v)g'(v)\cos u, f(v)g'(v)\sin u, -f(v)f'(v))\\
\|\mathbf{x}_u \wedge \mathbf{x}_v\| &= f(v)\sqrt{(f'(v))^2+(g'(v))^2} > 0,
\end{aligned}$$

so the differential map d$\mathbf{x}$ is one-to-one.

This regular surface S is called the *surface of revolution* (旋轉曲面).

Definition 11 (page 76)**.**

(a) The curve C is called the *generating curve* (生成曲線) of S.

(b) The z axis is the *rotation axis* (旋轉軸) of S.

(c) The circles described by the pints of C are called the *parallels* (平行線) of S.

(d) The various positions of C on S are called the *meridians* (子午線) of S.

2.3 Change of Parameters: Differential Functions on Surface

When we try to define a *smooth function* (可微分函數) $f : S \to \mathbb{R}$ at a point p on a regular surface S, a natural way to proceed is to choose a coordinate neighborhood of p with coordinates (u, v), and then we say that f is smooth at p if its expression in the coordinates (u, v) has continuous partial derivatives of all orders. However, the same point on S can belong to various coordinate neighborhoods or coordinate systems, so it is necessary to prove that the smooth property does *not* depend on the chosen coordinate systems.

Proposition 1 (Change of Parameters, page 70)**.** *Let p be a point on a regular surface S, and let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ and $\mathbf{y} : V \subset \mathbb{R}^2 \to S$ be two parameterizations of S such that $p \in \mathbf{x}(U) \cap \mathbf{y}(V) = W$. Then the change of coordinates map $h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$ is smooth and has a smooth inverse $h^{-1} = \mathbf{y}^{-1} \circ \mathbf{x} : \mathbf{x}^{-1}(W) \to \mathbf{y}^{-1}(W)$.*

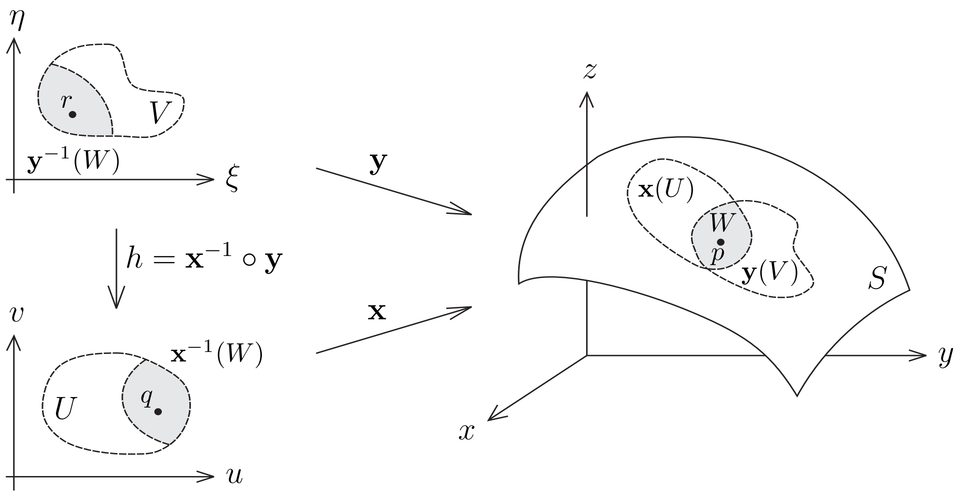

Figure 1: Change of parameters.

In other words, if $\mathbf{x}$ and $\mathbf{y}$ are given by

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U,$$
$$\mathbf{y}(\xi, \eta) = (x(\xi, \eta), y(\xi, \eta), z(\xi, \eta)), \quad (\xi, \eta) \in V,$$

then the change of coordinates map h, given by

$$u = u(\xi, \eta), \quad v = v(\xi, \eta), \quad (\xi, \eta) \in \mathbf{y}^{-1}(W)$$

are smooth functions, and the inverse map h^{-1}, given by

$$\xi = \xi(u, v), \quad \eta = \eta(u, v), \quad (u, v) \in \mathbf{x}^{-1}(W)$$

are also smooth. Remark that since

$$\frac{\partial(u, v)}{\partial(\xi, \eta)} \cdot \frac{\partial(\xi, \eta)}{\partial(u, v)} = 1,$$

this implies that the Jacobian determinants of both h and h^{-1} are nonzero everywhere.

Proof. First, we know that $h = \mathbf{x}^{-1} \circ \mathbf{y}$ is a homeomorphism since it is composed of homeomorphisms. It is *not* possible to conclude that h is differentiable by an analogous argument. This is because that $\mathbf{x}^{-1}$ is defined in an open subset of S, but we do *not* know what is meant by a differentiable function on S.

We proceed in the following way. Let $r \in \mathbf{y}^{-1}(W)$ and set $q = h(r)$. Since $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ is a parametrization, without loss of generality, we can assume that

$$\frac{\partial(x, y)}{\partial(u, v)}(q) \neq 0.$$

We extend the map $\mathbf{x}$ to a map $F : U \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$F(u, v, t) = (x(u, v), y(u, v), z(u, v) + t), \quad (u, v) \in U, \quad t \in \mathbb{R}.$$

Geometrically, F maps a vertical cylinder C over U into a "vertical cylinder" over $\mathbf{x}(U)$ by mapping each section of C with height t into the surface $\mathbf{x}(u, v)+t\mathbf{e}_3$, where $\mathbf{e}_3$ is the unit vector of the z axis.

We know that F is differentiable and the restriction $F|_{U\times\{0\}} = \mathbf{x}$. Furthermore, the determinant of the differential $\mathrm{d}F_q$ is

$$\det(\mathrm{d}F_q) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & 1 \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}(q) \neq 0.$$

By the inverse function theorem, there exists a neighborhood M of $\mathbf{x}(q)$ in $\mathbb{R}^3$ such that F^{-1} exists and is differentiable in M.

By the continuity of $\mathbf{y}$, there exists a neighborhood N of r in V such that $\mathbf{y}(N) \subset M$. Notice that, restricted to N, $h|_N = F^{-1} \circ \mathbf{y}|_N$ is a composition of differentiable maps. By the chain rule, we conclude that h is differentiable at r. Since r is arbitrary, h is differentiable on $\mathbf{y}^{-1}(W)$.

Exactly the same argument can be applied to show that the map h^{-1} is differentiable, and for higher order differentiability. □

Once we prove the change of parameter property, we will give an explicit definition of a smooth function on a regular surface.

Definition 2 (page 72)**.** Let $f : V \subset S \to \mathbb{R}$ be a function defined in an open subset V of a regular surface S. Then f is said to be *smooth* (光滑的) at $p \in V$ if, for *some* parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ with $p \in \mathbf{x}(U) \subset V$, the composition $f \circ \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ is smooth at $\mathbf{x}^{-1}(p)$. We say f is smooth in V if it is smooth at all points of V.

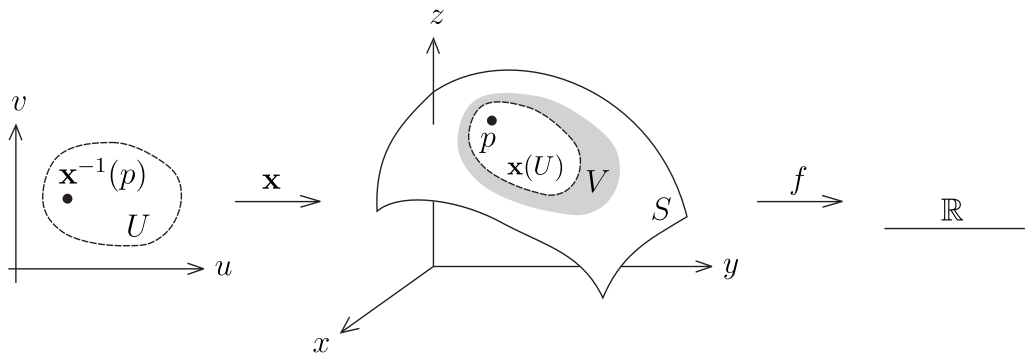

Figure 2: Define a smooth function on a regular surface.

From Proposition 1, the definition of a smooth function on a surface does *not* depend on the choice of the parametrization $\mathbf{x}$. In fact, if $\mathbf{y} : V \subset \mathbb{R}^2 \to S$ is another parametrization with $p \in \mathbf{y}(V)$, and if $h = \mathbf{x}^{-1} \circ \mathbf{y}$, then

$$f \circ \mathbf{y} = f \circ \mathbf{x} \circ \mathbf{x}^{-1} \circ \mathbf{y} = f \circ \mathbf{x} \circ (\mathbf{x}^{-1} \circ \mathbf{y}) = f \circ \mathbf{x} \circ h$$

is also smooth.

We often make the notational abuse of indicating f and $f \circ \mathbf{x}$ by the symbol $f(u, v)$, and say that $f(u, v)$ is the expression of f in the system of coordinates $\mathbf{x}$.

Example 3 (page 72)**.** Let S be a regular surface and $V \subset \mathbb{R}^3$ be an open set such that $S \subset V$. Let $f : V \subset \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. Then the restriction of f to S is a differentiable function on S. In fact, for any $p \in S$ and any parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ in p, the function $f \circ \mathbf{x} : U \to \mathbb{R}$ is differentiable. In particular, we often consider the following two functions.

(a) The *height function* (高度函數) relative to a unit vector $\mathbf{v} \in \mathbb{R}^3$ is given by $h : S \to \mathbb{R}$, $h(p) = \mathbf{p} \cdot \mathbf{v}$, where $\mathbf{p}$ is the position vector of $p \in S$.

(b) The square of the distance from a fixed point $p_0 \in \mathbb{R}^3$, $f(p) = \|p - p_0\|^2, p \in S$. The need for taking the square comes from the fact that the distance $\|p - p_0\|$ is *not* differentiable at $p = p_0$.

The definition of differentiability can be extended to mappings between surfaces.

Definition 4 (page 73)**.** A continuous map $\varphi : V \subset S_1 \to S_2$ of an open set V of a regular surface S_1 to a regular surface S_2 is said to be *differentiable* (可微分的) at $p \in V$ if given parameterizations

$$\mathbf{x}_1 : U_1 \subset \mathbb{R}^2 \to S_1, \qquad \mathbf{x}_2 : U_2 \subset \mathbb{R}^2 \to S_2,$$

with $p \in \mathbf{x}_1(U_1)$ and $\varphi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, the map

$$\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1 : U_1 \to U_2$$

is differentiable at $q = \mathbf{x}_1^{-1}(p)$.

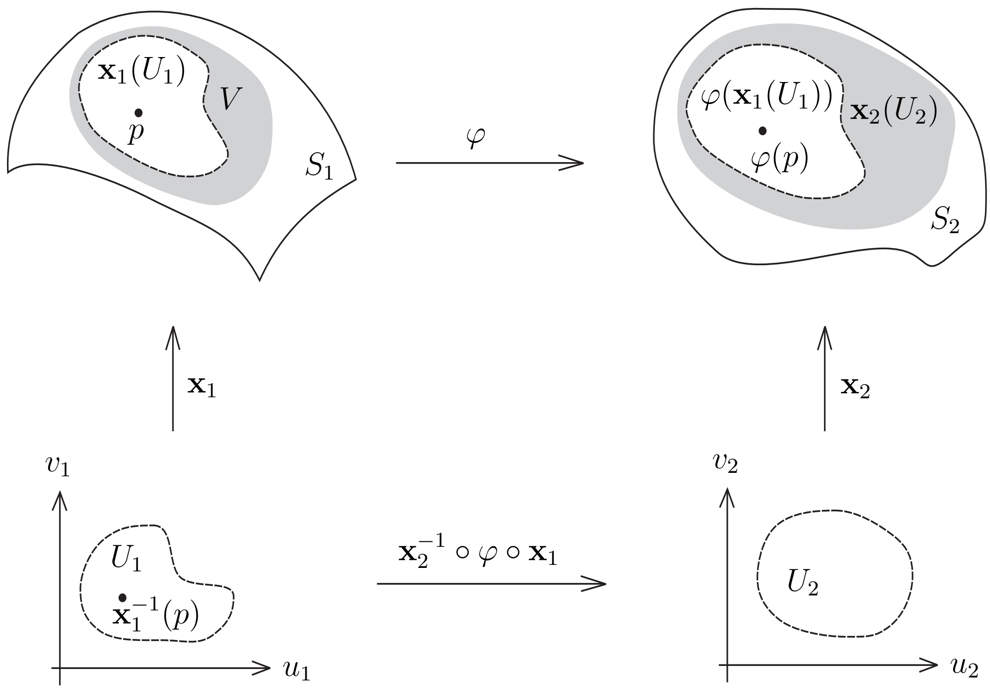

Figure 3: A differentiable function between to regular surfaces.

Definition 5 (page 74)**.** Two regular surface S_1 and S_2 are *diffeomorphic* (微分同胚的) if there exists a differentiable map $\varphi : S_1 \to S_2$ with a differentiable inverse $\varphi^{-1} : S_2 \to S_1$. Such a φ is called a *diffeomorphism* (微分同胚) from S_1 to S_2.

Example 6 (page 74)**.** If $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ is a parametrization, $\mathbf{x}^{-1} : \mathbf{x}(U) \to \mathbb{R}^2$ is differentiable. In fact, for any $p \in \mathbf{x}(U)$ and any parametrization $\mathbf{y} : V \subset \mathbb{R}^2 \to S$ in p, we have that $\mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$, where $W = \mathbf{x}(U) \cap \mathbf{y}(V)$ is differentiable. This shows that U and $\mathbf{x}(U)$ are diffeomorphic. That is, every regular surface is locally diffeomorphic to a plane.

2.4 The Tangent Plane; the Differential of a Map

In this section, we will show that condition (c) in the definition of a regular surface S guarantees that for every $p \in S$ the set of tangent vectors to the parameterized curves of S passing through p constitutes a plane.

Definition 1 (page 83)**.** By a *tangent vector* (切向量) to S at a point $p \in S$, we mean that the tangent vector $\boldsymbol{\alpha}'(0)$ of a differentiable parameterized curve $\boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to S$ with $\boldsymbol{\alpha}(0) = p$.

Proposition 2 (page 83)**.** *Let* $\mathbf{x} : U \subset \mathbb{R} \to S$ *be a parametrization of a regular surface* S *and let* $q \in U$*. The vector subspace of dimension* 2*,*

$$\mathrm{d}\mathbf{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3,$$

coincides with the set of tangent vectors to S *at* $\mathbf{x}(q)$*.*

Proof. (1) Given a tangent vector $\mathbf{w}$ to S at $\mathbf{x}(q)$, we want to find a curve $\boldsymbol{\beta}(t)$ on U with $\boldsymbol{\beta}(0) = q$ and $\boldsymbol{\beta}'(0) \in \mathbb{R}^2$ such that $\mathrm{d}\mathbf{x}_q(\boldsymbol{\beta}'(0)) = \mathbf{w}$.

Let $\mathbf{w}$ be a tangent vector to S at $\mathbf{x}(q)$. then $\mathbf{w} = \boldsymbol{\alpha}'(0)$, where $\boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to \mathbf{x}(U) \subset S$ is a differentiable parameterized curve and $\boldsymbol{\alpha}(0) = \mathbf{x}(q)$. Then the curve $\boldsymbol{\beta} = \mathbf{x}^{-1} \circ \boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to U \subset \mathbb{R}^2$ is a differentiable parameterized curve. By the definition of the differential, we have $\mathrm{d}\mathbf{x}_q(\boldsymbol{\beta}'(0)) = \mathbf{w}$. Hence $\mathbf{w} \in \mathrm{d}\mathbf{x}_q(\mathbb{R}^2)$.

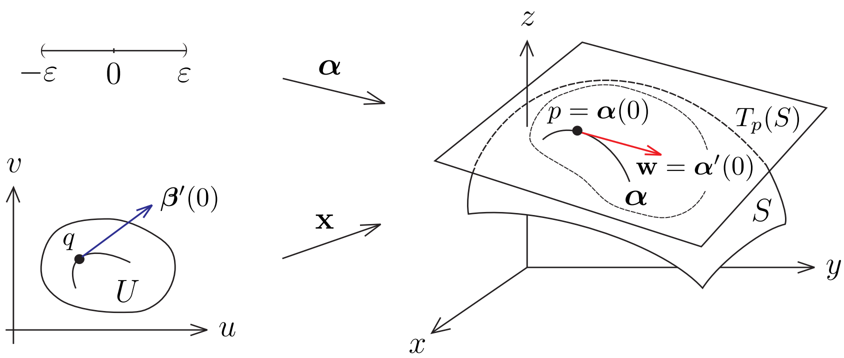

Figure 4: Tangent vector to a regular surface.

(2) On the other hand, let $\mathbf{w} = \mathrm{d}\mathbf{x}_q(\mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^2$. It is clear that $\mathbf{v}$ is the velocity vector of the curve $\boldsymbol{\gamma} : (-\varepsilon, \varepsilon) \to U$ given by

$$\boldsymbol{\gamma}(t) = t\mathbf{v} + q, t \in (-\varepsilon, \varepsilon).$$

By the definition of the differential, $\mathbf{w} = \boldsymbol{\alpha}'(0)$, where $\boldsymbol{\alpha} = \mathbf{x} \circ \boldsymbol{\gamma}$. This shows that $\mathbf{w}$ is a tangent vector. $\square$

By the above proposition, the plane $\mathrm{d}\mathbf{x}_q(\mathbb{R}^2)$, which passes through $\mathbf{x}(q) = p$, does *not* depend on the parametrization $\mathbf{x}$. This plane will be called the *tangent plane* (切平面) to S at p and will be denoted by $T_p(S)$. The choice of the parametrization $\mathbf{x}$ determines a basis $\{\frac{\partial \mathbf{x}}{\partial u}(q), \frac{\partial \mathbf{x}}{\partial v}(q)\}$ of $T_p(S)$, called the basis associated to $\mathbf{x}$. Sometimes it is convenient to write $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$.

The coordinates of a vector $\mathbf{w} \in T_p(S)$ in the basis $\{\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v}\}$ associated to a parametrization $\mathbf{x}$ are determined as follows. Since $\mathbf{w}$ is he velocity vector $\boldsymbol{\alpha}'(0)$ of a curve $\boldsymbol{\alpha} = \mathbf{x} \circ \boldsymbol{\beta}$, where $\boldsymbol{\beta} : (-\varepsilon, \varepsilon) \to U$ is given by $\boldsymbol{\beta}(t) = (u(t), v(t))$, with $\boldsymbol{\beta}(0) = q = \mathbf{x}^{-1}(p)$, we have

$$\boldsymbol{\alpha}'(0) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x} \circ \boldsymbol{\beta})(0) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(u(t), v(t))\bigg|_{t=0} = \mathbf{x}_u u'(0) + \mathbf{x}_v v'(0) = \mathbf{w}.$$

Thus, in the basis $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$, $\mathbf{w}$ has coordinates $(u'(0), v'(0))$. where $(u(t), v(t))$ is the expression, in the parametrization $\mathbf{x}$, of a curve whose velocity vector at $t = 0$ is $\mathbf{w}$.

With the notion of a tangent plane, we can talk about the differential of a smooth map between surfaces. Let S_1 and S_2 be two regular surfaces and let $\varphi : V \subset S_1 \to S_2$ be a differentiable mapping of an open set V of S_1 into S_2. If $p \in V$, we know that every tangent vector $\mathbf{w} \in T_p(S_1)$ is the velocity vector $\boldsymbol{\alpha}'(0)$ of a smooth parameterized curve $\boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to V$ with $\boldsymbol{\alpha}(0) = p$. The curve $\boldsymbol{\beta} = \varphi \circ \boldsymbol{\alpha}$ is such that $\boldsymbol{\alpha}(0) = \varphi(p)$, and therefore $\boldsymbol{\beta}'(0)$ is a vector of $T_{\varphi(p)}(S_2)$.

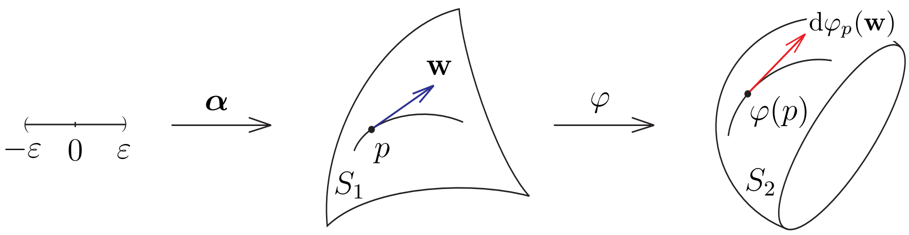

Figure 5: Differential of a smooth map between surfaces.

Proposition 3 (page 84)**.** *In the discussion above, given* $\mathbf{w}$*, the vector* $\boldsymbol{\alpha}'(0)$ *does not depend on the choice of* $\boldsymbol{\alpha}$*. The map* $\mathrm{d}\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2)$ *defined by* $\mathrm{d}\varphi_p(\mathbf{w}) = \boldsymbol{\beta}'(0)$ *is a linear transformation.*

Proof. Let $\mathbf{x}(u, v)$ and $\overline{\mathbf{x}}(\overline{u}, \overline{v})$ be parametrization in neighborhoods of p and $\varphi(p)$, respectively. Suppose that φ is expressed in these coordinates by

$$\varphi(u, v) = (\varphi_1(u, v), \varphi_2(u, v))$$

and that $\boldsymbol{\alpha}$ is expressed by

$$\boldsymbol{\alpha}(t) = (u(t), v(t)), \quad t \in (-\varepsilon, \varepsilon).$$

Then $\boldsymbol{\beta}(t) = (\varphi_1(u(t), v(t)), \varphi_2(u(t), v(t)))$, and the expression of $\boldsymbol{\beta}'(0)$ in the basis $\{\overline{\mathbf{x}}_u, \overline{\mathbf{x}}_v\}$ is

$$\boldsymbol{\beta}'(0) = \left(\frac{\partial \varphi_1}{\partial u} u'(0) + \frac{\partial \varphi_1}{\partial v} v'(0), \frac{\partial \varphi_2}{\partial u} u'(0) + \frac{\partial \varphi_2}{\partial v} v'(0) \right).$$

The relation above shows that $\boldsymbol{\beta}'(0)$ depends only on the map φ and the coordinates $(u'(0), v'(0))$ of $\mathbf{w}$ in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$. The vector $\boldsymbol{\beta}'(0)$ is therefore independent of $\boldsymbol{\alpha}$. Moreover, the same relation shows that

$$\boldsymbol{\beta}'(0) = \mathrm{d}\varphi_p(\mathbf{w}) = \begin{bmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_2}{\partial u} & \frac{\partial \varphi_2}{\partial v} \end{bmatrix} \begin{bmatrix} u'(0) \\ v'(0) \end{bmatrix}.$$

That is, $\mathrm{d}\varphi_p$ is a linear mapping of $T_p(S_1)$ into $T_{\varphi(p)}(S_2)$ whose matrix in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p(S_1)$ and $\{\overline{\mathbf{x}}_u, \overline{\mathbf{x}}_v\}$ of $T_{\varphi(p)}(S_2)$ is just the matrix given above. □

The linear map $\mathrm{d}\varphi_p$ defined by Proposition 3 is called the *differential* (微分映射) of φ at $p \in S_1$. In a similar way we defined the differential of a smooth function $f : U \to S \to \mathbb{R}$ at $p \in U$ as a linear map $\mathrm{d}f_p : T_p(S) \to \mathbb{R}$.

Example 4 (page 86)**.** Let $\mathbf{v} \in \mathbb{R}^3$ be a unit vector and let $h : S \to \mathbb{R}$, $h(p) = \mathbf{v} \cdot p, p \in S$, be the height function. To compute $\mathrm{d}h_p(\mathbf{w}), \mathbf{w} \in T_p(S)$, choose a differentiable curve $\boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to S$ with $\boldsymbol{\alpha}(0) = p, \boldsymbol{\alpha}'(0) = \mathbf{w}$. Since $h(\boldsymbol{\alpha}(t)) = \boldsymbol{\alpha}(t) \cdot \mathbf{v}$, we obtain

$$\mathrm{d}h_p(\mathbf{w}) = \frac{\mathrm{d}}{\mathrm{d}t} h(\boldsymbol{\alpha}(t)) \bigg|_{t=0} = \boldsymbol{\alpha}'(0) \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{v}.$$

Example 5 (page 86)**.** Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let $R_{z,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation of angle θ about the z axis. Then $R_{z,\theta}$ restricted to $\mathbb{S}^2$ is a differentiable map of $\mathbb{S}^2$. We shall compute $(\mathrm{d}R_{z,\theta})_p(\mathbf{w}), p \in \mathbb{S}^2, \boldsymbol{\alpha}'(0) = \mathbf{w}, \mathbf{w} \in T_p(\mathbb{S}^2)$.

Let $\boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to \mathbb{S}^2$ be a smooth curve with $\boldsymbol{\alpha}(0) = p, \boldsymbol{\alpha}'(0) = \mathbf{w}$. Then, since $R_{z,\theta}$ is linear,

$$(\mathrm{d}R_{z,\theta})_p(\mathbf{w}) = \frac{\mathrm{d}}{\mathrm{d}t} (R_{z,\theta} \circ \boldsymbol{\alpha}(t)) \bigg|_{t=0} = R_{z,\theta}(\boldsymbol{\alpha}'(0)) = R_{z,\theta}(\mathbf{w}).$$

Observe that $R_{z,\theta}$ leaves the north pole $N = (0, 0, 1)$ fixed, and that $(\mathrm{d}R_{z,\theta})_N : T_N(\mathbb{S}^2) \to T_N(\mathbb{S}^2)$ is just a rotation of angle θ in the plane $T_N(\mathbb{S}^2)$.

Given a point p on a regular surface S, there are two unit vectors of $\mathbb{R}^3$ that are normal to the tangent plane $T_p(S)$; each of them is called a *unit normal vector* (單位法向量) at p. The unit normal vector can be derived by

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|}(q) \quad \text{or} \quad -\frac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|}(q).$$

Definition 6 (page 87)**.**

(a) The straight line that passes through p and contains a unit normal vector at p is called the *normal line* (法線) at p.

(b) The *angle* (夾角) of two intersecting surfaces at an intersection point p is the angle of their tangent planes (or their normal lines) at p.

Example 7. Find the equations of the tangent plane and normal line at $P(-2, 1, 3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

Solution.

2.5 The First Fundamental Form

In this section we will study further geometric structures carried by the surface called the first fundamental form.

The natural inner product of $\mathbb{R}^3$ induces on each tangent plane $T_p(S)$ of a regular surface S an inner product $\langle \cdot, \cdot \rangle_p$. If $\mathbf{w}_1, \mathbf{w}_2 \in T_p(S) \subset \mathbb{R}^3$, then $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle_p$ is equal to the inner product of $\mathbf{w}_1$ and $\mathbf{w}_2$ as product in $\mathbb{R}^3$.

Definition 1 (page 92)**.** The quadratic form $I_p : T_p(S) \to \mathbb{R}$, defined by

$$I_p(\mathbf{w}) = \langle \mathbf{w}, \mathbf{w} \rangle = \|\mathbf{w}\|^2 \geq 0,$$

is called the *first fundamental form* (第一基本式) of the regular surface $S \subset \mathbb{R}^3$ at $p \in S$.

The first fundamental form is merely the expression of how the surface S inherits the natural inner product of $\mathbb{R}^3$. In the following paragraphs, we will see that the first fundamental form allows us to make measurements on the surface such as lengths of curves, angles of tangent vectors, areas of regions, without referring back to the ambient space $\mathbb{R}^3$ where the surface lies.

Now we will express the first fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ associated to a parametrization $\mathbf{x}(u, v)$ at p. Since a tangent vector $\mathbf{w} \in T_p(S)$ is the tangent vector to a parameterized curve $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t)), t \in (-\varepsilon, \varepsilon)$, with $\boldsymbol{\alpha}(0) = \mathbf{x}(u_0, v_0) = p$, we obtain

$$\begin{aligned}
I_p(\boldsymbol{\alpha}'(0)) &= \langle \boldsymbol{\alpha}'(0), \boldsymbol{\alpha}'(0) \rangle_p \\
&= \langle \mathbf{x}_u(u_0, v_0)u'(0) + \mathbf{x}_v(u_0, v_0)v'(0), \mathbf{x}_u(u_0, v_0)u'(0) + \mathbf{x}_v(u_0, v_0)v'(0) \rangle_p \\
&= \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p (u'(0))^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle_p u'(0)v'(0) + \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p (v'(0))^2 \\
&= E(u_0, v_0)(u'(0))^2 + 2F(u_0, v_0)u'(0)v'(0) + G(u_0, v_0)(v'(0))^2.
\end{aligned}$$

By letting p run in the coordinate neighborhood corresponding to $\mathbf{x}(u, v)$ we obtain functions

$$E(u, v) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \quad F(u, v) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad G(u, v) = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$$

which are differentiable in that neighborhood.

It is more common to represent the first fundamental form by the *differential form* (微分形式)

$$\mathrm{d}s^2 = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2.$$

Example 2 (page 93)**.** Find a coordinate system and the first fundamental form of the plane $P \subset \mathbb{R}^3$ passing through $p_0 = (x_0, y_0, z_0)$ and containing the orthonormal vectors $\mathbf{w}_1 = (a_1, a_2, a_3), \mathbf{w}_2 = (b_1, b_2, b_3)$.

Solution.

Example 3 (page 93)**.** Find a parametrization of the right cylinder over the circle $x^2 + y^2 = 1$ and its first fundamental form.

Solution.

☐ 雖然圓柱與平面在 $\mathbb{R}^3$ 當中的長相不同, 但是它們的第一基本式相同。

Example 4 (page 94)**.** Consider a helix that is given by $\boldsymbol{\alpha}(u) = (\cos u, \sin u, au)$. Through each point of the helix, draw a line parallel to the xy-plane and intersecting the z-axis. The surface generated by these lines is called a *helicoid* (螺旋面) and admits the following parametrization:

$$\mathbf{x}(u, v) = (v\cos u, v\sin u, au), \quad 0 < u < 2\pi, \quad v \in \mathbb{R}.$$

Compute the first fundamental form of the helix.

Solution.

Lengths of the curve and angles of tangent vectors

Definition 5 (page 95)**.** The *arc length* (弧長) s of a parameterized curve $\boldsymbol{\alpha} : I \to S$ with $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ is given by

$$s(t) = \int_{t_0}^{t} \|\boldsymbol{\alpha}'(\bullet)\| \,\mathrm{d}\bullet = \int_{t_0}^{t} \sqrt{I(\boldsymbol{\alpha}'(\bullet))}\,\mathrm{d}\bullet = \int_{t_0}^{t} \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\,\mathrm{d}\bullet\,.$$

Example 6. Let C be the curve on the cone $z = \sqrt{x^2 + y^2}$ whose projection onto the xy plane is the polar curve $r = \mathrm{e}^{\theta}, 0 \le \theta \le 1$. Find the length of the curve.

Solution.

Definition 7 (page 95)**.** The *angle* (夾角) θ under which two parameterized regular curves $\boldsymbol{\alpha} : I \to S, \boldsymbol{\beta} : I \to S$ intersect at $t = t_0$ is given by

$$\cos\theta = \frac{\langle \boldsymbol{\alpha}'(t_0), \boldsymbol{\beta}'(t_0)\rangle}{\|\boldsymbol{\alpha}'(t_0)\|\|\boldsymbol{\beta}'(t_0)\|}.$$

In particular, the angle φ of the coordinate curves of a parametrization $\mathbf{x}(u, v)$ is

$$\cos\varphi = \frac{\langle \mathbf{x}_u, \mathbf{x}_v\rangle}{\|\mathbf{x}_u\|\|\mathbf{x}_v\|} = \frac{F}{\sqrt{EG}}.$$

Definition 8 (page 95)**.** The coordinate curves of a parametrization are *orthogonal* (正交) if and only if $F(u, v) = 0$ for all u, v. Such a parametrization is called an *orthogonal parametrization* (正交參數化).

Example 9 (page 95)**.** Compute the first fundamental form of a sphere given by the parametrization

$$\mathbf{x}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

Solution.

Example 10 (page 96)**.** Find the curves in a neighborhood of the sphere which make a constant angle β with the meridians $\theta =$ constant. These curves are called *loxodromes* (rhumb lines, 斜駛線, 恒向線) of the sphere.

Solution.

Area of the surface, page 98

Definition 11 (page 98)**.** Let $R \subset S$ be a bounded region of a regular surface contained in the coordinate neighborhood of the parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$. The positive number

$$A(R) = \iint_Q \|\mathbf{x}_u \wedge \mathbf{x}_v\| \, \mathrm{d}u \, \mathrm{d}v, \quad Q = \mathbf{x}^{-1}(R),$$

is called the area (面積) of R.

We have to show that the integral does *not* depend on the parametrization. Suppose that $\overline{\mathbf{x}} : \overline{U} \subset \mathbb{R}^2 \to S$ be another parametrization with $R \subset \overline{\mathbf{x}}(\overline{U})$ and set $\overline{Q} = \overline{\mathbf{x}}^{-1}(R)$. Let $\frac{\partial(u,v)}{\partial(\overline{u},\overline{v})}$ be the Jacobian of the change of parameters $h = \mathbf{x}^{-1} \circ \overline{\mathbf{x}}$, then

$$\iint_{\overline{Q}} \|\overline{\mathbf{x}}_{\overline{u}} \wedge \overline{\mathbf{x}}_{\overline{v}}\| \, \mathrm{d}\overline{u} \, \mathrm{d}\overline{v} = \iint_{\overline{Q}} \|\mathbf{x}_u \wedge \mathbf{x}_v\| \left| \frac{\partial(u,v)}{\partial(\overline{u},\overline{v})} \right| \mathrm{d}\overline{u} \, \mathrm{d}\overline{v} = \iint_Q \|\mathbf{x}_u \wedge \mathbf{x}_v\| \, \mathrm{d}u \, \mathrm{d}v.$$

Since $\|\mathbf{x}_u \wedge \mathbf{x}_v\|^2 + \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2 = \|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2$, we know that the integrand of $A(R)$ can be written as

$$\|\mathbf{x}_u \wedge \mathbf{x}_v\| = \sqrt{\|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 - \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2} = \sqrt{EG - F^2}.$$

Example 12. Show that the surface area of the sphere with radius R is $4\pi R^2$.

Solution.

Example 13 (page 98)**.** Compute the area of the torus parameterized by

$$\mathbf{x}(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u), \quad 0 < u < 2\pi, 0 < v < 2\pi.$$

Solution.