Differential Geometry of Curves and Surfaces

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This lecture note we follow do Carmo's book "Differential geometry of curves and surfaces" to study the geometric objects in Euclidean space.

Chapter 1 Curves

1.1 Introduction

The differential geometry of curves and surfaces has two aspects. One is the study of local properties of curves and surfaces. That is, we will find the behavior of the curve or surfaces in the neighborhood of a point. The other is the study of global differential geometry, which will study the influence of the local properties on the behavior of the entire curve or surface.

1.2 Parameterized Curves

Our goal is to characterize certain subsets of $\mathbb{R}^3$ to be called space curves. A natural way of defining such subsets is through smooth vector-valued functions, which have continuous derivatives of all orders.

Definition 1 (page 2)**.**

(a) A *parameterized smooth curve* (可參數化的光滑曲線) is a smooth map (or vector-valued function) $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b) \subset \mathbb{R}$ into $\mathbb{R}^3$. We will write $\alpha(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$, where $t \in (a, b)$, and three *component functions* (分量函數) $x(t)$, $y(t)$, and $z(t)$ are smooth.

(b) The variable t is called *parameter* (參數) of the curve $\boldsymbol{\alpha}(t)$. The interval can be chosen in a general sense that $a = -\infty$ or $b = \infty$, or closed interval.

(c) The image set $\boldsymbol{\alpha}(I) \subset \mathbb{R}^3$ is called the *trace* (軌跡) of α.

(d) The vector $\alpha'(t) = (x'(t), y'(t), z'(t)) \in \mathbb{R}^3$ is called the *tangent vector* (切向量, or *velocity vector* 速度向量) of the curve α at t.

One should carefully distinguish a parameterized curve, which is a map, from its trace, which is a subset of $\mathbb{R}^3$. In do Carmo's book, the word "differentiable" is in fact "infinitely differentiable," and we use the terminology "smooth" in this note.

☐ 以物理 (質點運動) 的觀點來看, $\boldsymbol{\alpha}(t)$ 也稱爲位置向量 (position vector)。

Example 2.

(a) Find a map for the straight line joining two points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$.

(b) Find a map that represents the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $y+z=2$.

Solution.

Example 3 (page 2)**.** The parameterized smooth curve given by

$$\boldsymbol{\alpha}(t) = (a\cos t, a\sin t, bt), t \in \mathbb{R}$$

has as its trace in $\mathbb{R}^3$ a *helix* (螺旋線) of pitch $2\pi b$ on the cylinder $x^2+y^2=a^2$.

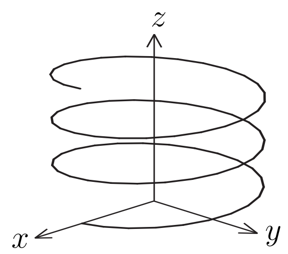

Figure 1: A helix $\boldsymbol{\alpha}(t) = (a\cos t, a\sin t, bt), t \in [0, 6\pi]$.

Example 4 (page 3)**.** The map $\boldsymbol{\alpha}: \mathbb{R} \to \mathbb{R}^2$ given by $\boldsymbol{\alpha}(t) = (t^3, t^2), t \in \mathbb{R}$, is a parameterized smooth plane curve. Notice that $\boldsymbol{\alpha}'(0) = (0,0)$; that is, the velocity vector is zero for $t=0$.

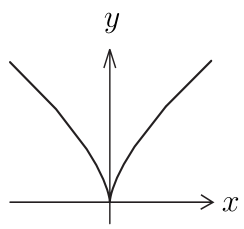

Figure 2: The map $\boldsymbol{\alpha}(t) = (t^3, t^2), t \in (-1, 1)$.

Example 5 (page 3)**.** The map $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ given by $\boldsymbol{\alpha}(t) = (t, |t|), t \in \mathbb{R}$ is not a parameterized smooth curve since $|t|$ is not differentiable at $t = 0$.

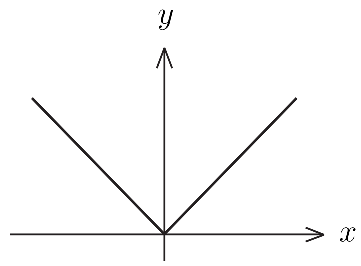

Figure 3: The map $\boldsymbol{\alpha}(t) = (t, |t|), t \in (-1, 1)$.

☐ 這述例子只是說明 $\boldsymbol{\alpha}(t) = (t, |t|)$ 不好, 然而 $\boldsymbol{\alpha}(I)$ 有可能存在光滑的參數化映射嗎?

Example 6 (page 3)**.** The map $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$ given by $(t^3 - 4t, t^2 - 4), t \in \mathbb{R}$, is a parameterized smooth curve. Notice that $\boldsymbol{\alpha}(2) = \boldsymbol{\alpha}(-2) = (0, 0)$; that is, the map $\boldsymbol{\alpha}$ is *not* one-to-one.

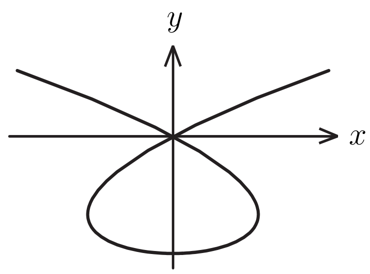

Figure 4: The map $\boldsymbol{\alpha}(t) = (t^3 - 4t, t^2 - 4), t \in (-2.5, 2.5)$.

☐ 曲線能否“好好地”放到空間中 — 稱爲嵌入 (embedded) — 是幾何上是重要的課題。

Example 7 (page 3)**.** Compare the following parameterized curves:

(a) $\boldsymbol{\alpha}_1(t) = (\cos t, \sin t), 0 \le t \le 2\pi$: ______________________________.

(b) $\boldsymbol{\alpha}_2(t) = (\cos t, \sin t), 0 \le t \le 4\pi$: ______________________________.

(c) $\boldsymbol{\alpha}_3(t) = (\cos 2t, \sin 2t), 0 \le t \le \pi$: ______________________________.

(d) $\boldsymbol{\alpha}_4(t) = (\sin t, \cos t), 0 \le t \le 2\pi$: ______________________________.

(e) $\boldsymbol{\alpha}_5(t) = (\cos t, -\sin t), 0 \le t \le 2\pi$: ______________________________.

☐ 同一曲線有很多參數法, 物理上稱爲 gauge invariance, 幾何上稱爲 diffeomorphism。

☐ 上述映射的軌跡都是圓, 它是很典型的封閉曲線 (closed curve) 的例子。

Here we list useful formulae of the vector-valued functions.

Proposition 8. *Suppose $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ are differentiable vector-valued functions, c is a real number, and $f(t)$ is a real-valued function. Then*

(a) $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\alpha}(t)+\boldsymbol{\beta}(t)) = \boldsymbol{\alpha}'(t)+\boldsymbol{\beta}'(t)$.

(b) $\frac{\mathrm{d}}{\mathrm{d}t}(c\boldsymbol{\alpha}(t)) = c\boldsymbol{\alpha}'(t)$.

(c) $\frac{\mathrm{d}}{\mathrm{d}t}(f(t)\boldsymbol{\alpha}(t)) = f'(t)\boldsymbol{\alpha}(t)+f(t)\boldsymbol{\alpha}'(t)$.

(d) $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\alpha}(t)\cdot\boldsymbol{\beta}(t)) = \boldsymbol{\alpha}'(t)\cdot\boldsymbol{\beta}(t)+\boldsymbol{\alpha}(t)\cdot\boldsymbol{\beta}'(t)$.

(e) $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\alpha}(t)\wedge\boldsymbol{\beta}(t)) = \boldsymbol{\alpha}'(t)\wedge\boldsymbol{\beta}(t)+\boldsymbol{\alpha}(t)\wedge\boldsymbol{\beta}'(t)$.

(f) $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\alpha}(f(t)) = \boldsymbol{\alpha}'(f(t))f'(t)$.

Example 9. If $\|\boldsymbol{\alpha}(t)\|^2 = c > 0$, where c is a constant, then $\boldsymbol{\alpha}(t)\cdot\boldsymbol{\alpha}(t) = c^2$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\alpha}(t)\cdot\boldsymbol{\alpha}(t)) =$$

Thus $\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}(t) = 0$, which says that ______________________.

1.3 Regular Curves; Arc Length

Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a parameterized smooth curve. For each $t \in I$ where $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$, there is a well-defined straight line, which contains the point $\boldsymbol{\alpha}(t)$ and the vector $\boldsymbol{\alpha}'(t)$. The line is called the *tangent line* (切線) to $\boldsymbol{\alpha}$ at t. For the study of the differential geometry of a curve it is essential that there exists such a tangent line at every point.

Definition 1 (page 6)**.** We call any point t where $\boldsymbol{\alpha}'(t) = \mathbf{0}$ a *singular point* (of order 0) (奇異點) of $\boldsymbol{\alpha}$. A parameterized smooth curve $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is said to be *regular* (正則的) if $\boldsymbol{\alpha}'(t) \neq 0$ for all $t \in I$.

In the following paragraphs, we will pay attention to regular parameterized smooth curves. Remark that the curve $\boldsymbol{\alpha}(t) = (x(t), y(t), z(t))$ is regular if and only if for all $t \in I$, at least one of $x'(t), y'(t), z'(t) \neq 0$.

Question. What is the "natural" or "good" parametrization for a space curve?

Definition 2 (page 6)**.** Given $t \in I$, the *arc length* (弧長) of a regular parameterized smooth curve $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ from the point t0 is defined by

$$s(t) = \int_{t_0}^{t} \|\boldsymbol{\alpha}'(u)\|\,\mathrm{d}u = \int_{t_0}^{t} \sqrt{(x'(u))^2+(y'(u))^2+(z'(u))^2}\,\mathrm{d}u.$$

Since $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$, the arc length function $s(t)$ is a differentiable function of t, and by the Fundamental Theorem of Calculus, we have

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \underline{\hspace{3cm}},$$

so we know that $s(t)$ is an increasing function, and thus a one-to-one function. It also implies that its inverse function $t(s)$ exists and

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{\frac{\mathrm{d}s}{\mathrm{d}t}},$$

is a continuous function. Hence for a regular parameterized smooth curve $\boldsymbol{\alpha}(t) = (x(t), y(t), z(t)), a \leq t \leq b$, it can be reparameterized by arc length function

$$\boldsymbol{\alpha}(s) = \boldsymbol{\alpha}(t(s)) = (x(t(s)), y(t(s)), z(t(s))), \quad c \leq s \leq d,$$

where $t(c) = a$ and $t(d) = b$. We will show that $\boldsymbol{\alpha}(s)$ is a regular parametrization:

$$\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} \Rightarrow \left\|\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}s}\right\| = \left\|\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\right\| \left|\frac{\mathrm{d}t}{\mathrm{d}s}\right| = \|\boldsymbol{\alpha}'(t)\| \cdot \frac{1}{\|\boldsymbol{\alpha}'(t)\|} \equiv 1.$$

so $\boldsymbol{\alpha}(s)$ is a nonzero, continuous vector-valued function.

Thus the arc length s can be introduced along the curve as a parameter. It is often useful to *parameterize a curve by arc length* (以弧長爲參數) because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system.

☐ 一條正則的可參數化光滑曲線, 它的弧長參數是「好的」且「自然的」參數表示法。

Example 3 (page 22)**.** Reparameterize the helix $\boldsymbol{\alpha}(t) = (a\cos t, a\sin t, bt)$ with respect to arc length measured from $(a, 0, 0)$ in the direction of increasing t. Find the length of the arc of the circular helix from the point $(a, 0, 0)$ to the point $(a, 0, 2\pi b)$.

Solution.

1.4 The Vector Product in $\mathbb{R}^3$

In this section, we will give a quick review about some properties of the vector product in $\mathbb{R}^3$.

Vector Product

Definition 4 (page 12)**.** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The *vector product* (or *cross product*, 外積) of $\mathbf{u}$ and $\mathbf{v}$ is the unique vector $\mathbf{u} \wedge \mathbf{v} \in \mathbb{R}^3$ defined by

$$(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$$

for all $\mathbf{w} \in \mathbb{R}^3$.

Let $\{\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)\}$ be a natural basis in $\mathbb{R}^3$. If

$$\mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e}_i, \quad \mathbf{v} = \sum_{i=1}^{3} v_i \mathbf{e}_i, \quad \text{and} \quad \mathbf{w} = \sum_{i=1}^{3} w_i \mathbf{e}_i,$$

then

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{e}_3.$$

Here we list some properties of the vector product:

(a) $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$. (anticommutativity).

(b) $(a\mathbf{u} + b \wedge \mathbf{v}) \wedge \mathbf{w} = a\mathbf{u} \wedge \mathbf{w} + b\mathbf{v} \wedge \mathbf{w}$ for any $a, b\mathbf{R}$. (linear)

(c) $\mathbf{u} \wedge \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

(d) $(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{u} = 0$; $(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{v} = 0$.

(e) If $\theta \in [0, \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\|\mathbf{u} \wedge \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$.

- Remark that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$.

The vector product of $\mathbf{u}$ and $\mathbf{v}$ is a vector $\mathbf{u} \wedge \mathbf{v}$ perpendicular to a plane generated by $\mathbf{u}$ and $\mathbf{v}$, with a norm to the area of a parallelogram generated by $\mathbf{u}$ and $\mathbf{v}$ and a direction such that $\{\mathbf{u}, \mathbf{v}, \mathbf{u} \wedge \mathbf{v}\}$ is a positive oriented basis (positive determinant).

The vector product is *not* associative. In fact, we have the following identity:

$$(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}.$$

Another formula is sometimes used:

$$(\mathbf{u} \wedge \mathbf{v}) \cdot (\mathbf{x} \wedge \mathbf{y}) = \begin{vmatrix} \mathbf{u} \cdot \mathbf{x} & \mathbf{v} \cdot \mathbf{x} \\ \mathbf{u} \cdot \mathbf{y} & \mathbf{v} \cdot \mathbf{y} \end{vmatrix}.$$

1.5 The Local Theory of Curves Parameterized by Arc Length

Question. How do we know that two space curves are the same (congruent)?

Space curves

Suppose that $\boldsymbol{\alpha}(s) : I \to \mathbb{R}^3$ is a regular parameterized smooth curve by arc length. We will first construct the moving frame to analyze the curve. In the following paragraphs, the parameter s stands for the arc length parameter, and the parameter t stands for arbitrary parameter.

Definition 1 (page 16, 18)**.**

(a) The *unit tangent vector* (單位切向量) is defined by $\mathbf{t}(s) = \boldsymbol{\alpha}'(s)$.

(b) The number $\kappa(s) = \|\boldsymbol{\alpha}''(s)\|$ is called the *curvature* (曲率) of $\boldsymbol{\alpha}$ at s.

Since $\mathbf{t}(s) = \boldsymbol{\alpha}'(s)$ has unit length, the norm $\|\boldsymbol{\alpha}''(s)\|$ measures the rate of change of the angle which neighboring tangents make with the tangent at s. So the curvature $\kappa(s)$ gives a measure of how rapidly the curve pulls away from the tangent line at s.

Example 2 (page 16)**.** If $\boldsymbol{\alpha}$ is a straight line, $\boldsymbol{\alpha}(s) = \mathbf{u}s + \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are constant vectors with $\|\mathbf{u}\| = 1$, then $\kappa \equiv 0$. Conversely, if $\kappa = \|\boldsymbol{\alpha}''(s)\| = 0$, then it implies $\boldsymbol{\alpha}''(s) = \mathbf{0}$, and by integration we get $\boldsymbol{\alpha}(s) = \mathbf{u}s + \mathbf{v}$ for some constant vectors $\mathbf{u}$ and $\mathbf{v}$. So the curve is a straight line.

At points where $\kappa(s) \neq 0$, a unit vector $\mathbf{n}(s)$ in the direction $\boldsymbol{\alpha}''(s)$ is well-defined by the equation $\mathbf{t}'(s) = \boldsymbol{\alpha}''(s) = \kappa(s)\mathbf{n}(s)$. Moreover, since $\boldsymbol{\alpha}'(s) \cdot \boldsymbol{\alpha}'(s) = 1$, we have $\boldsymbol{\alpha}''(s) \cdot \boldsymbol{\alpha}'(s) = 0$, so $\boldsymbol{\alpha}''(s)$ is normal to $\boldsymbol{\alpha}'(s)$.

At points where $\kappa(s) = 0$, the normal vector is *not* defined. We say that $s \in I$ is a *singular point of order* 1 (一階奇異點) if $\boldsymbol{\alpha}''(s) = \mathbf{0}$.

In the following paragraphs, we will restrict ourselves to curves smooth parameterized by arc length without singular points of order 1.

Definition 3 (page 17, 19)**.**

(a) The vector $\mathbf{n}(s)$ is called the *normal vector* (法向量) of $\boldsymbol{\alpha}(s)$ at s.

(b) The unit vector $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$ is called the *binormal vector* (or *principal normal* 次法向量) of $\boldsymbol{\alpha}(s)$ at s.

So for a parameterized curve $\boldsymbol{\alpha}(s)$, we have associated three orthonormal vectors $\mathbf{t}(s), \mathbf{n}(s)$, and $\mathbf{b}(s)$. The trihedron $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is called the *Frenet trihedron* (or *moving frame* 活動標架) of $\boldsymbol{\alpha}$ at s.

Definition 4 (page 19)**.** At some s, the plane determined by the unit tangent $\mathbf{t}$ and normal vectors $\mathbf{n}$ is called the *osculating plane* (密切平面). The plane spanned by $\mathbf{n}$ and $\mathbf{b}$ is called the *normal plane* (法平面). The plane spanned by $\mathbf{b}$ and $\mathbf{t}$ is called the *rectifying plane* (從切面).

Since $\mathbf{b}(s)$ is a unit vector, the length $\|\mathbf{b}'(s)\|$ measures the rate of change of the neighborhood osculating planes with the osculating plane at s; that is, $\mathbf{b}'(s)$ measures how rapidly the curve pulls away from the osculating plane at s. Since

$$\mathbf{b}'(s) = \mathbf{t}'(s) \wedge \mathbf{n}(s) + \mathbf{t}(s) \wedge \mathbf{n}'(s) = \mathbf{t}(s) \wedge \mathbf{n}'(s),$$

we know that $\mathbf{b}'(s)$ is normal to $\mathbf{t}(s)$. It follows that $\mathbf{b}'(s)$ is parallel to $\mathbf{n}(s)$. So we have the following definition.

Definition 5 (page 18)**.** The number $\tau(s)$ defined by $\mathbf{b}'(s) = \tau(s)\mathbf{n}(s)$ is called the *torsion* (扭率) of $\boldsymbol{\alpha}$ at s.

Remark that many authors write $\mathbf{b}'(s) = -\tau(s)\mathbf{n}(s)$ instead of our $\tau(s)$.

Example 6 (page 22)**.** Consider the helix, which is parameterized by

$$\boldsymbol{\alpha}(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right), \quad s \in \mathbb{R},$$

where $c^2 = a^2 + b^2$. Find the unit tangent, unit normal, and unit binormal vectors of the helix $\boldsymbol{\alpha}(s)$. Determine the curvature $\kappa(s)$ and the torsion $\tau(s)$ of $\boldsymbol{\alpha}(s)$.

Solution.

From the previous discussion, we have $\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s)$ and $\mathbf{b}'(s) = \tau(s)\mathbf{n}(s)$. Now we will derive the equation of $\mathbf{n}'(s)$. Since $\mathbf{n}(s) = \mathbf{b}(s) \wedge \mathbf{t}(s)$, we have

$$\begin{aligned}\mathbf{n}'(s) &= \mathbf{b}'(s) \wedge \mathbf{t}(s) + \mathbf{b}(s) \wedge \mathbf{t}'(s) = \tau(s)\mathbf{n}(s) \wedge \mathbf{t}(s) + \mathbf{b}(s) \wedge (\kappa(s)\mathbf{n}(s)) \\ &= -\tau(s)\mathbf{t}(s) \wedge \mathbf{n}(s) - \kappa(s)\mathbf{n}(s) \wedge \mathbf{b}(s) = -\kappa(s)\mathbf{t}(s) - \tau(s)\mathbf{b}(s).\end{aligned}$$

Thus, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & -\tau\\ 0 & \tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}. \tag{1}$$

Remark that the matrix is anti-symmetric.

Physically, we can think of a curve in $\mathbb{R}^3$ as being obtained from a straight line by bending (curvature) and twisting (torsion). After reflecting on this construction, we are led to state the following theorem, which shows that the curvature and torsion describe completely the local behavior of the curve.

Theorem 7 (Fundamental theorem of the local theory of curves, page 19)**.**

(a) (Existence) *Given smooth functions $\kappa(s) > 0$ and $\tau(s), s \in I$, there exists a regular parameterized smooth curve $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ such that s is the arc length, $\kappa(s)$ is the curvature, and $\tau(s)$ is the torsion of $\boldsymbol{\alpha}$.*

(b) (Rigidity) *Any other curve $\overline{\boldsymbol{\alpha}}$, satisfying the same condition, differs from $\boldsymbol{\alpha}$ by a rigid motion; that is, there exists an orthogonal linear map ρ of $\mathbb{R}^3$, with positive determinant, and a vector $\mathbf{c}$, such that $\overline{\boldsymbol{\alpha}} = \rho \circ \boldsymbol{\alpha} + \mathbf{c}$.*

We will not go through details about the proof of this theorem here. You may find the proof of the existence part in page 309–311 and the rigidity part in page 20–21 in do Carmo's book. However, the key ideas are the followings:

(a) First, from (1) we can prove that the moving frame $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is orthonormal for all s. Next, when we write down the equation (1) in terms of component functions, it can be viewed as the initial valued problem of the first order linear differential system in $I \times \mathbb{R}^9$. The existence and uniqueness of the linear differential system is the standard result of the differential equation theory.

(b) For the rigidity part, we will show that two frames $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ and $\{\overline{\mathbf{t}}(s), \overline{\mathbf{n}}(s), \overline{\mathbf{b}}(s)\}$ are the same by checking

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\|\mathbf{t}(s) - \overline{\mathbf{t}}(s)\|^2 + \|\mathbf{n}(s) - \overline{\mathbf{n}}(s)\|^2 + \|\mathbf{b}(s) - \overline{\mathbf{b}}(s)\|^2\right) = 0$$

if $\mathbf{t}(s_0) = \overline{\mathbf{t}}(s_0), \mathbf{n}(s_0) = \overline{\mathbf{n}}(s_0)$, and $\mathbf{b}(s_0) = \overline{\mathbf{b}}(s_0)$.

☐ 我們可以利用曲率和扭率判斷兩空間曲線是否全等。

Compute the curvature and torsion in arbitrary parameter

Suppose that $\boldsymbol{\alpha}(t) : I \to \mathbb{R}^3$ is a regular (of order 1) parameterized smooth curve. We can compute the curvature and torsion of $\boldsymbol{\alpha}(t)$ in terms of t. That is, it is *not* necessary to reparameterize the curve $\boldsymbol{\alpha}(t(s))$ by arc length and then to compute geometric quantities. In fact, for general curves, it is very hard to get the arc length parameter explicitly, so it is much convenient when we derive the following formulae.

Theorem 8 (page 25)**.** *For a regular parameterized smooth curve* $\boldsymbol{\alpha}(t)$*,*

(a) *the curvature of* $\boldsymbol{\alpha}(t)$ *is*

$$\kappa(t) = \frac{\|\mathbf{t}'(t)\|}{\|\boldsymbol{\alpha}'(t)\|} = \frac{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|}{\|\boldsymbol{\alpha}'(t)\|^3}.$$

(b) *the torsion of* $\boldsymbol{\alpha}(t)$ *is*

$$\tau(t) = -\frac{(\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)) \cdot \boldsymbol{\alpha}'''(t)}{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|^2} = -\frac{\det(\boldsymbol{\alpha}'(t), \boldsymbol{\alpha}''(t), \boldsymbol{\alpha}'''(t))}{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|^2}.$$

Proof.

(a) By the Chain Rule, we have

$$\kappa \overset{\text{def.}}{=} \left\|\frac{\mathrm{d}\mathbf{t}(s)}{\mathrm{d}s}\right\| = \left\|\frac{\mathrm{d}\mathbf{t}(t(s))}{\mathrm{d}s}\right\| =$$

Since $\boldsymbol{\alpha}'(t) = \mathbf{t}(t)\|\boldsymbol{\alpha}'(t)\|$, we have

$$\boldsymbol{\alpha}''(t) =$$

and

$$\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t) =$$

Notice that $\|\mathbf{t}(t)\| = 1$, and it implies $\mathbf{t}(t)$ and $\mathbf{t}'(t)$ are orthogonal, so

$$\frac{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|}{\|\boldsymbol{\alpha}'(t)\|^3} =$$

(b) From the relation

$$\mathbf{t}(t) = \frac{\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|}$$

$$\mathbf{b}(t) = \frac{\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)}{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|}$$

$$\mathbf{n}(t) = \mathbf{b}(t) \wedge \mathbf{t}(t) = \frac{\|\boldsymbol{\alpha}'(t)\|}{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|}\,\boldsymbol{\alpha}''(t) - \frac{\boldsymbol{\alpha}'(t) \cdot \boldsymbol{\alpha}''(t)}{\|\boldsymbol{\alpha}'(t)\|\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|}\,\boldsymbol{\alpha}'(t),$$

by the Chain Rule, we get $\mathbf{b}'(t) = \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}$, which means

$$\tau(t)\,\mathbf{n}(t)\,\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}'''(t)}{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|}\right)(\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)).$$

It gives

$$\begin{aligned}\tau(t)\,\frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{(\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}'''(t)) \cdot \mathbf{n}(t)}{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|}\right)(\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)) \cdot \mathbf{n}(t) \\ &= -\frac{\|\boldsymbol{\alpha}'(t)\|(\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)) \cdot \boldsymbol{\alpha}'''(t)}{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|^2}\end{aligned}$$

Since $\frac{\mathrm{d}s}{\mathrm{d}t} = \|\boldsymbol{\alpha}'(t)\|$, we have

$$\tau(t) = -\frac{(\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)) \cdot \boldsymbol{\alpha}'''(t)}{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|^2} = -\frac{\det(\boldsymbol{\alpha}'(t), \boldsymbol{\alpha}''(t), \boldsymbol{\alpha}'''(t))}{\|\boldsymbol{\alpha}'(t) \wedge \boldsymbol{\alpha}''(t)\|^2}.$$

□

Example 9. Discuss the formulae of $\kappa(t)$ and $\tau(t)$.

Solution.

Orientation, page 11–12 and 16–17

Given the curve $\boldsymbol{\alpha}$ parameterized by arc length $s \in (a, b)$, we may consider the curve $\boldsymbol{\beta}$ defined in $(-b, -a)$ by $\boldsymbol{\beta}(-s) = \boldsymbol{\alpha}(s)$, which has the same trace as $\boldsymbol{\alpha}$ but is described in the opposite direction. In this case, we say two curves differ by a *change of orientation* (改變定向).

Notice that by a change of orientation, the tangent vector changes its direction; that is, if $\boldsymbol{\beta}(-s) = \boldsymbol{\alpha}(s)$, then

$$\frac{\mathrm{d}\boldsymbol{\beta}(-s)}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{\alpha}(s)}{\mathrm{d}s} \Rightarrow \frac{\mathrm{d}\boldsymbol{\beta}(-s)}{\mathrm{d}(-s)}\frac{\mathrm{d}(-s)}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{\alpha}(s)}{\mathrm{d}s} \Rightarrow \frac{\mathrm{d}\boldsymbol{\beta}(-s)}{\mathrm{d}(-s)} = -\frac{\mathrm{d}\boldsymbol{\alpha}(s)}{\mathrm{d}s}.$$

We can take derivative again and get the normal vector and the curvature are invariant under a change of orientation.

Since $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$, we know that the binormal vector changes its direction by a change of orientation. One can show that the torsion is invariant under a change of orientation.

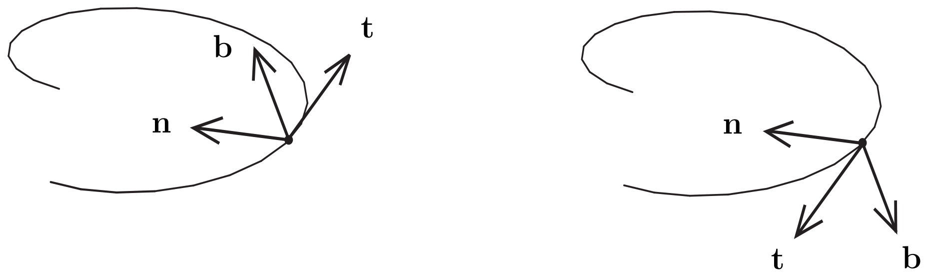

Figure 1: Directions of Frenet trihedron under a change of orientation.

In Euclidean space, we can also discuss the orientation property. Two ordered bases $[\mathbf{e}] = \{\mathbf{e}_i\}_{i=1}^n$ and $[\mathbf{f}] = \{\mathbf{f}_i\}_{i=1}^n$ of an n-dimensional vector space V have the *same orientation* (相同定向) if the matrix of change of basis has positive determinant. We denote this relation by $[\mathbf{e}] \sim [\mathbf{f}]$. From elementary properties of determinants, it follows that $[\mathbf{e}] \sim [\mathbf{f}]$ is an *equivalent relation* (等價關係); that is,

(a) $[\mathbf{e}] \sim [\mathbf{e}]$.

(b) If $[\mathbf{e}] \sim [\mathbf{f}]$, then $[\mathbf{f}] \sim [\mathbf{e}]$.

(c) If $[\mathbf{e}] \sim [\mathbf{f}]$ and $[\mathbf{f}] \sim [\mathbf{g}]$, then $[\mathbf{e}] \sim [\mathbf{g}]$.

The set of all ordered bases of V is decomposed into equivalent classes. Since the determinant of a change of basis is either positive or negative, there are only two equivalent classes. Each of the equivalent classes determined by the above relation is called an *orientation* (定向) of V. Therefore, V has two orientations, and if we fix one orientation, the other is called the *opposite orientation* (相反的定向).

Example 10 (page 12)**.** In the case $V = \mathbb{R}^3$, there exists a natural ordered basis $\{\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)\}$, and we will call the orientation corresponding to this basis the *positive orientation* (正的定向) of $\mathbb{R}^3$, and the other one being the *negative orientation* (負的定向).

Plane curves and signed curvature, page 21

For the plane curve $\boldsymbol{\alpha}(s) : I \to \mathbb{R}^2$, it is possible to give the curvature $\kappa(s)$ a sign. Let $\{\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)\}$ be the natural ordered basis of $\mathbb{R}^2$. We define the normal vector $\mathbf{n}(s), s \in I$, by requiring the basis $\{\mathbf{t}(s), \mathbf{n}(s)\}$ to have the same orientation as the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. In this setting, the curvature $\kappa(s)$ is defined by

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} \overset{\text{def.}}{=} \kappa(s)\mathbf{n}(s),$$

and $\kappa(s)$ might be either positive or negative.

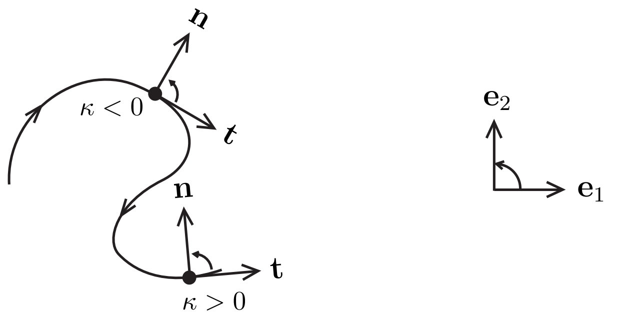

Figure 2: Signed curvature.

☐ 平面曲線可利用 $\mathbf{t}(s)$ 逆時針轉 $\frac{\pi}{2}$ 先定義 $\mathbf{n}(s)$, 再定義有符號的曲率。

One can show that $|\kappa|$ agrees with the previous definition and that κ changes sign when we change either the orientation of $\boldsymbol{\alpha}$ or the orientation of $\mathbb{R}^2$.

Example 11. Discuss the curvature of a circle with radius R.

Solution.

☐ 對於封閉曲線, 我們通常會選取曲線的定向使得法向量指向內部, 稱爲「正的定向」。

Example 12 (page 18)**.** If $\boldsymbol{\alpha}(s)$ is a plane curve, then the plane of the curve agrees with the osculating plane; hence $\tau(s) \equiv 0$. Conversely, if $\tau(s) \equiv 0$ and $\kappa(s) \neq 0$, then $\mathbf{b}(s) = \mathbf{b}$ is a well-defined unit vector independent of s. Since $\mathbf{t} \perp \mathbf{b}$, it gives

$$(\boldsymbol{\alpha}(s) \cdot \mathbf{b})' = \boldsymbol{\alpha}'(s) \cdot \mathbf{b} = 0,$$

and it implies that $\boldsymbol{\alpha}(s) \cdot \mathbf{b}$ is constant. Hence, $\boldsymbol{\alpha}(s)$ is contained in a plane normal to $\mathbf{b}$. Remark that the condition $\kappa(s) \neq 0$ everywhere is essential in this argument.

1.6 The Local Canonical Form

One of the effective methods of solving problems in geometry consists of finding a coordinate system which is adapted to the problem. In the study of local properties of a curve, we have a natural coordinate system called the Frenet trihedron.

Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a regular smooth curve parameterized by arc length s. We will write the vector valued function of the curve in a neighborhood of $s = 0$ using the trihedron $\{\mathbf{t}(0), \mathbf{n}(0), \mathbf{b}(0)\}$ as a basis for $\mathbb{R}^3$.

By the Taylor expansion

$$\boldsymbol{\alpha}(s) = \boldsymbol{\alpha}(0) + \frac{\boldsymbol{\alpha}'(0)}{1!}s + \frac{\boldsymbol{\alpha}''(0)}{2!}s^2 + \frac{\boldsymbol{\alpha}'''(0)}{3!}s^3 + \boldsymbol{R}(s),$$

where $\boldsymbol{R}(s)$ is the remainder satisfying $\lim\limits_{s\to 0} \frac{\boldsymbol{R}(s)}{s^3} = \mathbf{0}$, since $\boldsymbol{\alpha}'(0) = \mathbf{t}(0), \boldsymbol{\alpha}''(0) = \kappa(0)\mathbf{n}(0)$, and

$$\begin{aligned}
\boldsymbol{\alpha}'''(0) &= (\kappa \mathbf{n})'(0) = \kappa'(0)\mathbf{n}(0) + \kappa(0)\mathbf{n}'(0) \\
&= \kappa'(0)\mathbf{n}(0) + \kappa(0)(-\kappa(0)\mathbf{t}(0) - \tau(0)\mathbf{b}(0)) \\
&= -\kappa^2(0)\mathbf{t} + \kappa'(0)\mathbf{n}(0) - \kappa(0)\tau(0)\mathbf{b}(0),
\end{aligned}$$

we have

$$\begin{aligned}
&\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(0) \\
&= \left(s - \frac{\kappa^2(0)}{3!}s^3\right)\mathbf{t}(0) + \left(\frac{\kappa(0)}{2!}s^2 + \frac{\kappa'(0)}{3!}s^3\right)\mathbf{n}(0) - \frac{\kappa(0)\tau(0)}{3!}s^3\mathbf{b}(0) + \boldsymbol{R}(s).
\end{aligned}$$

Take the system $Oxyz$ such that the origin O agrees with $\boldsymbol{\alpha}(0)$ and that $\mathbf{t}(0) = (1,0,0)$, $\mathbf{n}(0) = (0,1,0)$, $\mathbf{b}(0) = (0,0,1)$. So $\boldsymbol{\alpha}(s) = (x(s), y(s), z(s))$, where

$$\begin{cases}
x(s) = s - \dfrac{\kappa^2(0)}{3!}s^3 + R_x(s) \\
y(s) = \dfrac{\kappa(0)}{2!}s^2 + \dfrac{\kappa'(0)}{3!}s^3 + R_y(s) \\
z(s) = -\dfrac{\kappa(0)\tau(0)}{3!}s^3 + R_z(s),
\end{cases} \tag{2}$$

This representation is called the *local canonical form* (局部自然表示法) of $\boldsymbol{\alpha}(s)$ in a neighborhood of $s = 0$.

Example 1 (page 28)**.** We can use the local canonical form to interpret of the geometric property of the sign of the torsion. From the third equation of (2), if $\tau(0) < 0$ and s is sufficiently small, then $z(s)$ increases with s. Let the "positive side" of the osculating plane be the side toward which $\mathbf{b}$ is pointing. Since $z(0) = 0$, when we describe the curve in the direction of increasing arc length, the curve will cross the osculating plane at $s = 0$ pointing toward the positive side.

If $\tau(0) > 0$, the curve described in the direction of increasing arc length will cross the osculating plane at $s = 0$ pointing to the side opposite the positive side. See Figure 3.

Example 2 (page 29)**.** Another application of the local canonical form is the existence of a neighborhood $J \subset I$ of $s = 0$ such that $\boldsymbol{\alpha}(J)$ is entirely contained in the one side of the rectifying plane toward which the vector $\mathbf{n}$ is pointing. This is because $\kappa(0) > 0$, we know that $y(s) \geq 0$ for s sufficiently small, and $y(s) = 0$ if and only if $s = 0$.

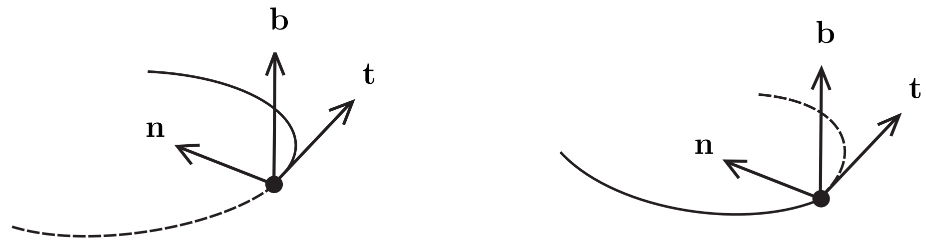

Figure 3: Negative torsion (left) and positive torsion (right). Curves are locally lie in the one side of the rectifying plane toward which the vector $\mathbf{n}$ is pointing.

Example 3 (page 29)**.** As third application of the canonical form, we will prove the following geometric meaning of the osculating plane:

> The osculating plane at s is the limit position of the plane determined by the tangent line at s and the point $\boldsymbol{\alpha}(s + h)$ when $h \to 0$.

To prove this, without loss of generality, we assume that $s = 0$. Thus, every plane containing the tangent line at $s = 0$ is of the form $z = cy$ or $y = 0$. We can viewed the plane $y = 0$ as the case $z = cy, c = \infty$. The condition for the plane $z = cy$ to pass through $\boldsymbol{\alpha}(0 + h)$ is

$$c(h) = \frac{z(h)}{y(h)} = \frac{-\frac{\kappa\tau}{6}h^3 + \cdots}{\frac{\kappa}{2}h^2 + \frac{\kappa^2}{6}h^3 + \cdots}.$$

Let $h \to 0$, we see that $c(h) \to 0$. Therefore, the limit position of the plane $z(h) = c(h)y(h)$ is the plane $z = 0$, that is, the osculating plane.

1.7 Global Properties of Plane Curves

In this section we want to describe two results of the global differential geometry of curves: (A) isoperimetric inequality and (B) the theorem of turning tangents. We skip the four-vertex theorem and Cauchy-Crofton formula in this lector note.

Definition 1 (page 30)**.** A *closed* (封閉的) plane curve is a regular parameterized smooth curve $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^2$ such that $\boldsymbol{\alpha}$ and all its derivatives agree at a and b; that is,

$$\boldsymbol{\alpha}_+^{(n)}(a) = \boldsymbol{\alpha}_-^{(n)}(b) \quad \text{for all} \quad n = 0, 1, 2, \ldots.$$

Definition 2 (page 30)**.** The curve $\boldsymbol{\alpha}$ is *simple* (簡單的) if it has no further self-intersection; that is, if $t_1, t_2 \in [a, b), t_1 \neq t_2$, then $\boldsymbol{\alpha}(t_1) \neq \boldsymbol{\alpha}(t_2)$.

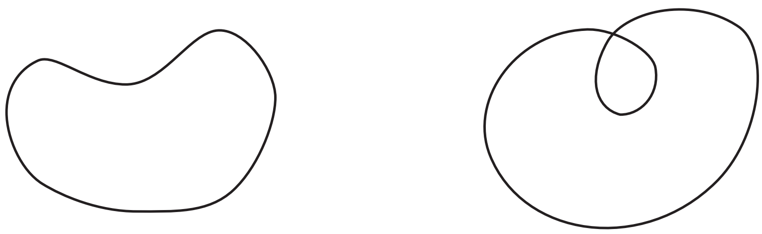

Figure 4: A simple closed curve and a nonsimple closed curve.

Definition 3 (page 31)**.** We say a simple closed curve is *positively oriented* (正的定向) if we choose a parametrization $\boldsymbol{\alpha}$ such that the normal vector is pointing the interior of the curve.

A. The Isoperimetric Inequality

Question. Of all simple closed curves in the plane with a given length l, which one bounds the largest area?

Recall that the area A bounded by a positively oriented simple closed curve $\boldsymbol{\alpha}(t) = (x(t), y(t))$, where $t \in [a, b]$ is an arbitrary parameter:

$$A = -\int_a^b y(t)x'(t)\,\mathrm{d}t = \int_a^b x(t)y'(t)\,\mathrm{d}t = \frac{1}{2}\int_a^b (x(t)y'(t) - y(t)x'(t))\,\mathrm{d}t.$$

Theorem 4 (The Isoperimetric Inequality, page 33)**.** *Let C be a simple closed plane curve with length l, and let A be the area of the region bounded by C. Then*

$$A \leq \frac{l^2}{4\pi},$$

and equality holds if and only if C is a circle.

Proof. First we choose two parallel tangent lines to C, called L and L', so that the curve is entirely contained in the strip bounded by L and L'. Consider a circle $\mathbb{S}^1$ which is tangent to both L and L' and does not meet C. Let O be the center of $\mathbb{S}^1$ and take a coordinate system with origin at O and the x-axis perpendicular to L and L' as Figure 5.

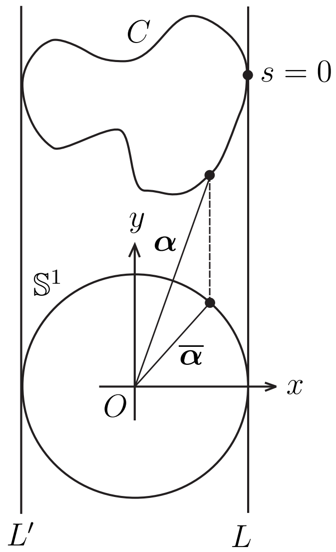

Figure 5: Proof of isoperimetric inequality.

We parameterize C by arc length, say $\boldsymbol{\alpha}(s) = (x(s), y(s))$, so that it is positively oriented and the tangency point of L is $s = 0$. We can assume that the parametrization of $\mathbb{S}^1$ is

$$\overline{\boldsymbol{\alpha}}(s) = (\overline{x}(s), \overline{y}(s)) \overset{\text{def.}}{=} (x(s), \overline{y}(s)), \quad s \in [0, l],$$

Let A and $\overline{A}$ be the area bounded by C and $\mathbb{S}^1$, respectively, then

$$A = \int_0^l x(s)y'(s)\mathrm{d}s, \quad \text{and} \quad \overline{A} = -\int_0^l \overline{y}(s)x'(s)\,\mathrm{d}s = \frac{\overline{l}^2}{4\pi},$$

where $\overline{l}$ is the circumference of $\mathbb{S}^1$ so that the radius of $\mathbb{S}^1$ is $\frac{\overline{l}}{2\pi}$. By the Cauchy inequality, we have

$$\begin{aligned} A + \frac{\overline{l}^2}{4\pi} = A + \overline{A} &= \int_0^l (x(s)y'(s) - \overline{y}(s)x'(s))\,\mathrm{d}s \\ &\le \int_0^l \sqrt{((x(s))^2 + (\overline{y}(s))^2)((x'(s))^2 + (y'(s))^2)}\,\mathrm{d}s \\ &= \int_0^l \sqrt{(\overline{x}(s))^2 + (\overline{y}(s))^2}\,\mathrm{d}s = \frac{l\overline{l}}{2\pi}, \end{aligned}$$

and it gives

$$A \leq -\frac{\bar{l}^2}{4\pi} + \frac{l\bar{l}}{2\pi} = -\frac{1}{4\pi}\left(\bar{l}^2 - 2l\bar{l} + l^2\right) + \frac{l^2}{4\pi} = -\frac{1}{4\pi}\left(\bar{l} - l\right)^2 + \frac{l^2}{4\pi} \leq \frac{l^2}{4\pi}.$$

Now, we consider the equality case. We know that $\bar{l} = l$, and two vectors $(x(s), -\overline{y}(s))$ and $(y'(s), x'(s))$ are parallel, which implies

$$x(x)x'(s) = -\overline{y}(s)y'(s). \tag{3}$$

Let $r = \frac{\bar{l}}{2\pi} = \frac{l}{2\pi}$ be the radius of $\mathbb{S}^2$ parameterized by $\overline{\boldsymbol{\alpha}}(s) = (x(s), \overline{y}(s))$, then we have $(x(s))^2 + (\overline{y}(s))^2 = r^2$, or $\overline{y}(s) = \pm\sqrt{r^2 - (x(s))^2}$. So equation (3) becomes

$$y'(s) = \mp\frac{x(s)x'(s)}{\sqrt{r^2 - (x(s))^2}}.$$

We can solve the differential equation directly:

$$\begin{aligned} y(s) &= \mp\int \frac{x(s)x'(s)}{\sqrt{r^2 - (x(s))^2}}\,\mathrm{d}s = \pm\frac{1}{2}\int \frac{1}{\sqrt{r^2 - (x(s))^2}}\,\mathrm{d}(r^2 - (x(s))^2) \\ &= \pm\sqrt{r^2 - (x(s))^2} + C_0. \end{aligned}$$

Since $\boldsymbol{\alpha}(0) = (x(0), y(0)) = (r, 0)$, we get $C_0 = 0$. Therefore, we have $(y(s))^2 = r^2 - (x(s))^2$, or $(x(s))^2 + (y(s))^2 = r^2$. The curve C is a circle. □

The Theorem of Turning Tangents

Let $\boldsymbol{\alpha} : [0, l] \to \mathbb{R}^2$ be a plane closed curve given by $\boldsymbol{\alpha}(s) = (x(s), y(s))$, where s is the arc length parameter. Then the tangent vector $\mathbf{t}(s) = (x'(s), y'(s))$ has unit length. It is convenient to introduce the *tangent indicatrix* (切線指標線) $\mathbf{t} : [0, l] \to \mathbb{R}^2$ by $\mathbf{t}(s) = (x'(s), y'(s))$. The tangent indicatrix is a smooth curve and the trace is contained in a circle of radius 1.

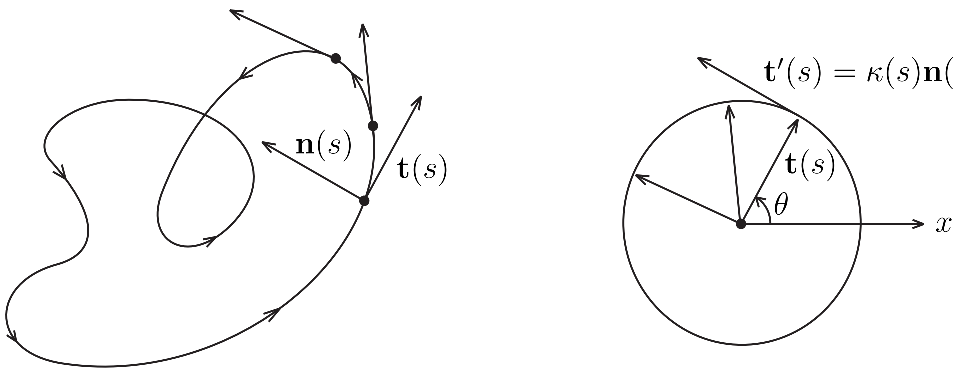

Figure 6: Tangent indicatrix.

Let $\theta(s)$ be the angle that $\mathbf{t}(s)$ makes with the x-axis. We can write

$$\mathbf{t}(s) = (x'(s), y'(s)) = (\cos\theta(s), \sin\theta(s)).$$

By the Chain Rule, we get

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}(\cos\theta(s), \sin\theta(s)) = (-\sin\theta(s)\theta'(s), \cos\theta(s)\theta'(s)) = \theta'(s)\mathbf{n}(s).$$

Since $\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa(s)\mathbf{n}(s)$, we have $\theta'(s) = \kappa(s)$, and we can rewrite $\theta(s) : [0, l] \to \mathbb{R}$ as

$$\theta(s) = \int_0^s \kappa(s)\,\mathrm{d}s.$$

Intuitively, $\theta(s)$ measures the total rotation of the tangent vector, that is, the total angle described by the point $\mathbf{t}(s)$ on the tangent indicatrix, as we run the curve $\boldsymbol{\alpha}$ from 0 to s. Since $\boldsymbol{\alpha}$ is closed, this angle is an integer multiple I of 2π; that is,

$$\int_0^l \kappa(s)\,\mathrm{d}s = \theta(l) - \theta(0) = 2\pi I.$$

The integer I is called the *rotation index* (旋轉指標) of the curve $\boldsymbol{\alpha}$. Notice that the rotation index changes sign when we change the orientation of the curve.

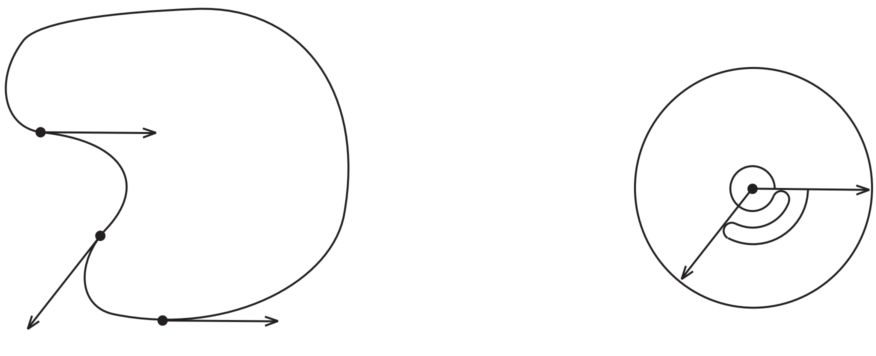

Figure 7: Rotation index $I = 1$.

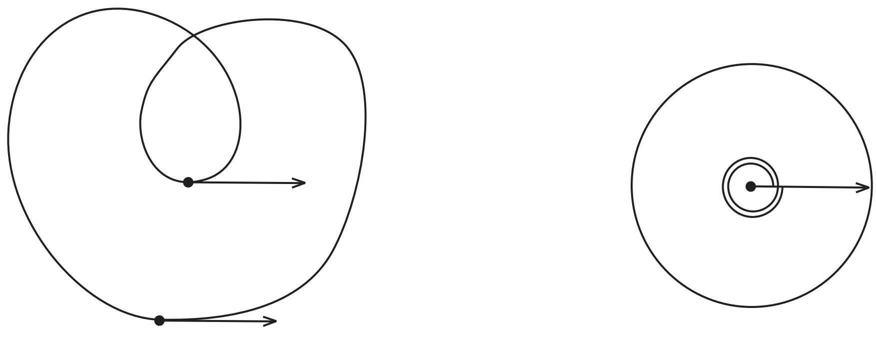

Figure 8: Rotation index $I = 2$.

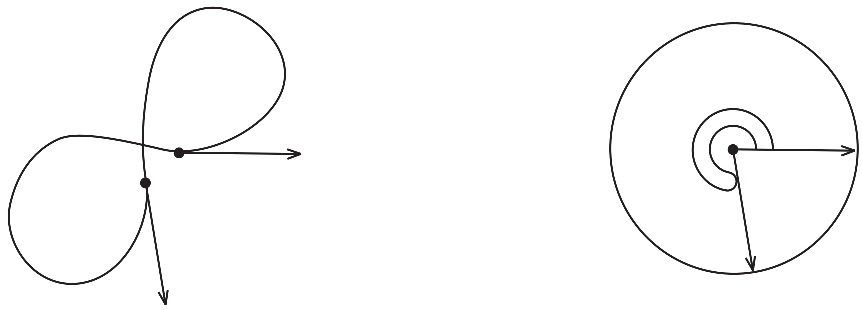

Figure 9: Rotation index $I = 0$.

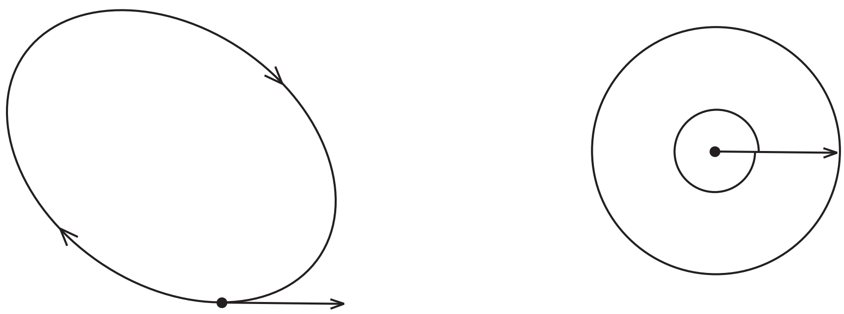

Figure 10: Rotation index $I = -1$.

An important global fact about the rotation index is the following theorem.

Theorem 5 (The theorem of turning tangents, page 37)**.** *The rotation index of a simple closed curve is* ± 1*, where the sign depends on the orientation of the curve.*