17.3 Applications of Second-Order Differential Equations, page 1168

Second-order differential equations have a variety of applications in science and engineering. In this section we explore two of them: the vibration of springs and electric circuits.

Vibrating Springs, page 1168

We consider the motion of an object with mass m at the end of a spring that is either vertical or horizontal on a level surface. Hooke's Law says that if the spring is stretched or compressed x units from its normal length, then it exerts a force that is proportional to x:

$$\text{resorting force} = -kx,$$

where k is a positive constant called the *spring constant* (彈簧常數). If we ignore any external resisting forces (due to air resistance or friction), then by Newton's Second Law, we have

$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx \quad \text{or} \quad m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = 0.$$

This is a second-order linear differential equation. Its characteristic equation is $mr^2 + k = 0$ with roots $r = \pm\sqrt{\frac{k}{m}}i = \pm\omega i$. Thus the general solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t = A \cos(\omega t + \delta),$$

where $\omega = \sqrt{\frac{k}{m}}$ is called frequency (頻率), $A = \sqrt{c_1^2 + c_2^2}$ is the amplitude (振幅), δ is the phase angle (相位角). This type of motion is called *simple harmonic motion* (簡諧運動).

Damped Vibrations, page 1169

We consider the motion of a spring that is subject to a frictional force (in the case of the horizontal spring) or a damping force (in the case where a vertical spring moves through a fluid).

We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. Thus

$$\text{damping force } = -c\,\frac{\mathrm{d}x}{\mathrm{d}t},$$

where c is a positive constant, called the *damping constant* (阻尼常數). Thus, in this case, Newton's Second Law gives

$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -c\,\frac{\mathrm{d}x}{\mathrm{d}t} - kx \quad \text{or} \quad m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\,\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0.$$

This is a second-order linear differential equation. The characteristic equation is $mr^2 + cr + k = 0$. The roots are

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}.$$

(a) If $c^2 - 4mk > 0$ (overdamping), then r_1 and r_2 are distinct real roots and $x = c_1\mathrm{e}^{r_1t} + c_2\mathrm{e}^{r_2t}$. Since c, m, and k are all positive, we have $\sqrt{c^2 - 4mk} < c$, so r_1 and r_2 must both be negative. This shows that $x \to 0$ as $t \to \infty$. Notice that oscillations do not occur. (It's possible for the mass to pass through the equilibrium position once, but only once.) This is because $c^2 > 4mk$ means that there is a strong damping force (high-viscosity oil or grease) compared with a weak spring or small mass.

(b) If $c^2 - 4mk = 0$ (critical damping), This case corresponds to equal roots $r_1 = r_2 = -\frac{c}{2m}$ and the solution is given by $x = (c_1 + c_2t)\mathrm{e}^{-\frac{c}{2m}t}$. It is similar to case (a), but the damping is just sufficient to suppress vibrations. Any decrease in the velocity of the fluid leads to the vibrations of the following case.

(c) If $c^2 - 4mk < 0$ (underdamping), we have $r_1 = -\frac{c}{2m} + \omega i$ and $r_2 = -\frac{c}{2m} - \omega i$, where $\omega = \frac{\sqrt{4mk-c^2}}{2m}$. The solution is given by $x = \mathrm{e}^{-\frac{c}{2m}t}(c_1\cos\omega t + c_2\sin\omega t)$. We see that there are oscillations that are damped by the factor $\mathrm{e}^{-\frac{c}{2m}t}$. Since $c > 0$ and $m > 0$, we have $-\frac{c}{2m} < 0$ so $\mathrm{e}^{-\frac{c}{2m}t} \to 0$ as $t \to \infty$. This implies that $x \to 0$ as $t \to \infty$; that is, the motion decays to 0 as time increases.

Forced Vibrations, page 1171

Suppose that in addition to the restoring force and the damping force, the motion of the spring is affected by an external force $F(t)$. Then Newton's Second Law gives

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F(t).$$

A commonly type of external force is periodic force function

$$F(t) = F_0\cos\omega_0 t \quad \text{where} \quad \omega_0 \neq \omega = \sqrt{\frac{k}{m}}.$$

In this case, and in the absence of a damping force ($c = 0$), we can use the method of undetermined coefficients to get

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t.$$

If $\omega_0 = \omega$, then the applied frequency reinforces the natural frequency and the the result is vibrations of large amplitude. This is the phenomenon of resonance (共振). The motion of the mass is given by

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2m\omega} t \sin \omega t.$$

Electric circuits, page 1172

We are in a position to analyze the circuit. It contains an electromotive force E (supplied by a battery or generator), a resistor R, an inductor L, and a capacitor C (電容器), in series. If the charge on the capacitor at time t is $Q = Q(t)$, then the current is the rate of change of Q with respect to t is $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$. The voltage drops across the resister, inductor, and capacitor are $RI, L\frac{\mathrm{d}I}{\mathrm{d}t}$, and $\frac{Q}{C}$, respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + RI + \frac{Q}{C} = E(t).$$

Since $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$, this equation becomes

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = E(t), \tag{1}$$

which is a second-order linear differential equation with constant coefficient. If the charge Q_0 and the current I_0 are known at time 0, then we have the initial conditions $Q(0) = Q_0$ and $Q'(0) = I(0) = I_0$. This initial-value problem can be solved.

A differential equation for the current can be obtained by differentiating (1) with respect to t and remembering that $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$:

$$L\frac{\mathrm{d}^2I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C} I = E'(t).$$