17.2 Nonhomogeneous Linear Equations, page 1160

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$ay'' + by' + cy = G(x), \tag{1}$$

where a, b, and c are constants and $G(x)$ is a continuous function.

Theorem 1 (page 1161)**.** *The general solution of the nonhomogeneous differential equation* (1) *can be written as* $y(x) = y_h(x) + y_p(x)$*, where* $y_h(x)$ *is the general homogeneous solution and* $y_p(x)$ *is a particular solution of equation* (1)*.*

Proof. If $y(x)$ is any solution of (1), then

$$a(y - y_p)'' + b(y - y_p)' + c(y - y_p) = (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) = 0.$$

It shows that $y - y_p$ is the solution of its homogeneous equation, and we write it as $y - y_p = y_h$, so $y(x) = y_h(x) + y_p(x)$. Conversely, if $y(x) = y_h(x) + y_p(x)$, then

$$a(y_h + y_p)'' + b(y_h + y_p)' + c(y_h + y_p) = (ay_h'' + by_h' + cy_h) + (ay_p'' + by_p' + cy_p) = G.$$

□

There are two methods for finding a particular solution: the method of undetermined coefficients and the method of variation of parameters.

The Method of Undetermined Coefficients (未定係數法)

Since exponential function, polynomials, and trigonometric functions $\sin mx, \cos mx$ are algebraiclly closed (代數封閉), if the nonhomogeneous terms consist of those functions, we can assume the particular solution y_p is of these form as well.

Summary of the method of undetermined coefficients:

(1) If $G(x) = \sum_{i=0}^{n} a_i x^i$, then we try $y_p(x) = x^j \sum_{i=0}^{n} A_i x^i$.

(2) If $G(x) = \mathrm{e}^{kx} \sum_{i=0}^{n} a_i x^i$, then we try $y_p(x) = \mathrm{e}^{kx} x^j \sum_{i=0}^{n} A_i x^i$.

(3) If $G(x) = \mathrm{e}^{kx} \cos mx \sum_{i=0}^{n} a_i x^i$ or $G(x) = \mathrm{e}^{kx} \sin mx \sum_{i=0}^{n} a_i x^i$, then we try $y_p(x) = x^j \mathrm{e}^{kx} \left(\cos mx \sum_{i=0}^{n} A_i x^i + \sin mx \sum_{i=0}^{n} B_i x^i \right)$.

- Here j is the smallest nonnegative integers $j = 0, 1, 2$ that will ensure that no term in the assumption is a solution of the corresponding homogeneous equation.

Remark 2. The method of undetermined coefficients is straightforward but works only for a restricted class of functions $G(x)$.

Example 1. Solve $y'' - 4y = \mathrm{e}^x \cos x$ by the method of undetermined coefficients..

Solution.

Example 2. Solve $y'' - y = x\mathrm{e}^x, y(0) = 2, y'(0) = 1$ by the method of undetermined coefficients.

Solution.

Example 3. Solve $y'' + y' - 2y = x + \sin 2x, y(0) = 1, y'(0) = 0$ by the method of undetermined coefficients.

Solution.

The Method of Variation of Parameters (參數變動法)

Suppose we have already solved the homogeneous equation $ay'' + by' + cy = 0$ and written the solution as

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Let $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ be a particular solution of the nonhomogeneous equation $ay'' + by' + cy = G(x)$. Then we have

$$\begin{aligned}
c \cdot y_p &= (u_1 y_1 + u_2 y_2) \cdot c \\
b \cdot y_p' &= (u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2') \cdot b \\
a \cdot y_p'' &= (u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2'') \cdot a
\end{aligned}$$

So

$$\begin{aligned}
ay_p'' + by_p' + cy_p &= a(u_1'' y_1 + 2u_1' y_1' + u_2'' y_2 + 2u_2' y_2') + b(u_1' y_1 + u_2' y_2) \\
&= a(u_1'' y_1 + u_1' y_1' + u_2'' y_2 + u_2' y_2') + a(u_1' y_1' + u_2' y_2') + b(u_1' y_1 + u_2' y_2).
\end{aligned}$$

Suppose that $u_1' y_1 + u_2' y_2 = F(x)$, then $u_1'' y_1 + u_1' y_1' + u_2'' y_2 + u_2' y_2' = F'(x)$. So $ay_p'' + by_p' + cy_p = G(x)$ becomes

$$\begin{cases} y_1 u_1' + y_2 u_2' = F(x) \\ y_1' u_1' + y_2' u_2' = \frac{G(x) - aF'(x) - bF(x)}{a} \end{cases} \tag{2}$$

We can solve u_1' and u_2' by (2). The function $F(x)$ gives us a freedom to get the particular solution of the nonhomogeneous equation. In particular, we can set $F(x) \equiv 0$ so that the system will be

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = \frac{G(x)}{a} \end{cases}.$$

Solutions $u_1'(x)$ and $u_2'(x)$ are

$$\begin{aligned}
u_1'(x) &= \frac{-\frac{G(x)}{a} y_2(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} \\
u_2'(x) &= \frac{\frac{G(x)}{a} y_1(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}.
\end{aligned}$$

Thus we can get $u_1(x)$ and $u_2(x)$ after integrating the above functions.

Remark 3. The method of variation of parameters works for every continuous function $G(x)$ but is usually more difficult to apply in practice.

Example 4. Solve the equation $y'' + y = \tan x$, $0 < x < \frac{\pi}{2}$.

Solution.

Example 5. Solve the equation $y'' + 3y' + 2y = \sin(\mathrm{e}^x)$ by the method of variation of parameters.

Solution.