Chapter 17 Second-Order Differential Equations

17.1 Second-Order Linear Equation (page 1154)

A *second-order linear differential equation* (二階線性微分方程) has the form

$$P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)y(x) = G(x), \tag{1}$$

where P, Q, R, and G are continuous functions.

Definition 1 (page 1154)**.**

(a) If $G(x) = 0$ for all x, we say (1) *homogeneous linear equations* (齊次線性方程).

(b) If $G(x) \neq 0$ for some x, we say (1) *nonhomogeneous linear equations* (非齊次線性方程).

In this section, we will focus on the solutions of the homogeneous linear equation:

$$P(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + Q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)y(x) = 0. \tag{2}$$

Theorem 1 (page 1154)**.** *If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation (2) and c_1 and c_2 are any constants, then the function $y(x) = c_1y_1(x) + c_2y_2(x)$ is also a solution of Equation (2).*

Proof. For $i = 1, 2$, we have $P(x)y_i'' + Q(x)y_i' + R(x)y_i = 0$. So

$$\begin{aligned}
&P(x)y'' + Q(x)y' + R(x)y \\
&= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\
&= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\
&= c_1(P(x)y_1'' + Q(x)y_1' + R(x)y_1) + c_2(P(x)y_2'' + Q(x)y_2' + R(x)y_2) \\
&= c_1 \cdot 0 + c_2 \cdot 2 = 0.
\end{aligned}$$

□

Definition 2 (page 1155)**.** Two continuous functions y_1 and y_2 are called *linearly independent* (線性獨立) if neither y_1 nor y_2 is a constant multiple of the other.

Theorem 2 (page 1155)**.** *If y_1 and y_2 are linearly independent solutions of (2) on an interval, and $P(x)$ is never 0, then the general solution is given by $y(x) = c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are arbitrary constants.*

Here we will discuss the second-order homogeneous linear differential equation with constant coefficients, that is P, Q, and R are constant functions. In this case, we write the differential equation as

$$ay''(x) + by' + cy(x) = 0, \tag{3}$$

where a, b, and c are constants and $a \neq 0$.

Consider $y = \mathrm{e}^{rx}$, where r is a constant, then $y' = r\mathrm{e}^{rx}$ and $y'' = r^2\mathrm{e}^{rx}$. So

$$ar^2\mathrm{e}^{rx} + br\mathrm{e}^{rx} + c\mathrm{e}^{rx} = 0 \Rightarrow (ar^2 + br + c)\mathrm{e}^{rx} = 0 \Rightarrow ar^2 + br + c = 0.$$

Thus $y = \mathrm{e}^{rx}$ is a solution of (3) if r is a root of

$$ar^2 + br + c = 0. \tag{4}$$

Equation (4) is called the *auxiliary equation* (or *characteristic equation*) (輔助方程、特徵方程) of the differential equation $ay'' + by' + cy = 0$.

There are three cases for the roots of the equation (4).

(1) <u>If $b^2 - 4ac > 0$</u>, then $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ are two real distinct roots. So $y_1 = \mathrm{e}^{r_1 x}$ and $y_2 = \mathrm{e}^{r_2 x}$ are two linearly independent solutions. Hence the general solution of $ay'' + by' + cy = 0$ is $y = c_1\mathrm{e}^{r_1 x} + c_2\mathrm{e}^{r_2 x}$.

(2) <u>If $b^2 - 4ac = 0$</u>, then $r_1 = r_2 = -\frac{b}{2a}$ are two real double roots. Here we denote the double roots by r. So $y_1 = \mathrm{e}^{rx}$ is a solution of $ay'' + by' + cy = 0$. To find another solution y_2 which is linearly independent of y_1, we consider the method of *reduction of order* (降階法). Let $y_2 = u(x)y_1(x)$, where $u(x) \neq$ constant function, be another solution of $ay'' + by' + cy = 0$. Since

$$\begin{aligned} c \cdot y_2 &= c \cdot uy_1 \\ b \cdot y_2' &= b \cdot (u'y_1 + uy_1') \\ a \cdot y_2'' &= a \cdot (u''y_1 + 2u'y_1' + uy_1''), \end{aligned}$$

we have

$$ay_2'' + by_2' + cy_2 = au''y_1 + u'(2ay_1' + by_1) + u(ay_1'' + by_1' + cy_1) = au''y_1 = 0,$$

and it implies $u''(x) = 0$, $u'(x) = C_1$, and $u(x) = C_1 x + C_2$. In particular, $u(x) = x$ is a candidate, and the general solution of solution of $ay''+by'+cy = 0$ is $y = c_1\mathrm{e}^{rx} + c_2 x\mathrm{e}^{rx}$.

(3) <u>If $b^2-4ac\le 0$</u>, then $r_1=\frac{-b+\sqrt{4ac-b^2}\,i}{2a}$ and $r_2=\frac{-b-\sqrt{4ac-b^2}\,i}{2a}$ are two conjugate complex roots. Denote $\alpha=-\frac{b}{2a}$ and $\beta=\frac{\sqrt{4ac-b^2}}{2a}$. When we consider the equation and solution in the complex sense, which is a algebraic closed field, we write the solution of the differential equation as

$$\begin{aligned} y &= C_1\mathrm{e}^{r_1x}+C_2\mathrm{e}^{r_2x}=C_1\mathrm{e}^{(\alpha+\beta i)x}+C_2\mathrm{e}^{(\alpha-\beta i)x} \\ &= C_1\mathrm{e}^{\alpha x}(\cos\beta x+i\sin\beta x)+C_2\mathrm{e}^{\alpha x}(\cos(-\beta x)+i\sin(-\beta x)) \\ &= (C_1+C_2)\mathrm{e}^{\alpha x}\cos\beta x+i(C_1-C_2)\mathrm{e}^{\alpha x}\sin\beta x \end{aligned}$$

The solution is real if and only if $c_1=C_1+C_2$ and $c_2=i(C_1-C_2)$ are real numbers, that is, $y=\mathrm{e}^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ is the general solution.

Initial-Value and Boundary-Value Problems, page 1158

An *initial-value problem* (初始值問題) for the second-order linear equation consists of finding a solution y of the differential equation that also satisfies initial conditions of the form $y(x_0)=y_0, y'(x_0)=y_1$, where y_0 and y_1 are given constants. If $P,Q,R,$ and G are continuous on an interval and $P(x)\neq 0$ there, then a theorem guarantees the existence and uniqueness of a solution to this initial-value problem.

A *boundary-value problem* (邊界值問題) for the second-order linear equation consists of finding a solution y of the differential equation that also satisfies boundary conditions of the form $y(x_0)=y_0, y(x_1)=y_1$. In contrast with the situation for initial-value problems, a boundary-value problem does *not* always have a solution.

Example 1 (page 1156, 1158)**.** Solve the initial value problem $y''+y'-6y=0$, $y(0)=1$ and $y'(0)=1$.

Solution.

Example 2 (page 1157)**.** Solve the boundary value problem $y''+2y'+y=0, y(0)=1, y(1)=3$.

Solution.

Example 3 (page 1158)**.** Solve the equation $y'' - 6y' + 13y = 0$.

Solution.

Example 4 (page 1158)**.** Solve the initial-value problem $y''+y = 0, y(0) = 2, y'(0) = 3$.

Solution.

Example 5. We can view $ay'' + by' + cy = 0$ as $\left(\frac{\mathrm{d}}{\mathrm{d}x} + r_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + r_2\right) y = 0$, where r_1 and r_2 are two roots of the corresponding characteristic equation. Let $Y = \left(\frac{\mathrm{d}}{\mathrm{d}x} + r_2\right) y$. Then we can solve $\left(\frac{\mathrm{d}}{\mathrm{d}x} + r_1\right) Y = 0$, and then we solve $\left(\frac{\mathrm{d}}{\mathrm{d}x} + r_2\right) y = Y$ by the integrating factor method. In this case, you will easily see that the solutions are “2-dimensional”.