16.9 The Divergence Theorem, page 1141

In section 16.5, we have discussed the vector version of the following line integral

$$\int_C \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s = \iint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{d}A = \iint_D \operatorname{div}\mathbf{F}\,\mathrm{d}A,$$

where C is the positively oriented boundary curve of the plane region D. Here we will generalize this result to vector fields on $\mathbb{R}^3$, and this is called the *Divergence Theorem* (散度定理).

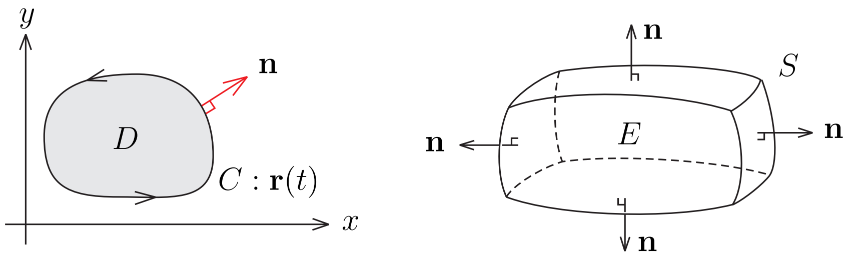

Figure 1: Normal line integral (left) and the Divergence Theorem (right).

The Divergence Theorem (page 1141)**.** *Let E be a simple solid region and let S be the boundary surface of E, given with positive outward orientation. Let* $\mathbf{F}$ *be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then*

$$\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,\mathrm{d}V.$$

The Divergence Theorem relates the integral of a derivative of a function ($\operatorname{div}\mathbf{F}$) over a region to the integral of the integral of the origin function $\mathbf{F}$ over the boundary of the region.

Example 1 (page 1143)**.** Find the flux of the vector field $\mathbf{F}(x,y,z) = z\,\mathbf{i}+y\,\mathbf{j}+x\,\mathbf{k}$ over the unit sphere $x^2+y^2+z^2=1$.

Solution.

Example 2 (page 1143)**.** Evaluate $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + (y^2 + \mathrm{e}^{xz^2})\,\mathbf{j} + \sin(xy)\,\mathbf{k},$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0, y = 0$, and $y + z = 2$.

Solution.

Example 3. Let

$$\mathbf{F}(x, y, z) = \left(xy^2 + \sqrt{y^2 + z^4}\right)\mathbf{i} + \left(\tan^{-1} x + x^2 y\right)\mathbf{j} + \left(\frac{z^3}{3} - \mathrm{e}^{x^2+y^2}\right)\mathbf{k}.$$

Find $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where the surface S is the top half of the sphere $x^2 + y^2 + z^2 = 1$ with the unit normal vectors pointing away from the origin.

Solution.

Example 4. Let S be the sphere $x^2+y^2+z^2=1$.

(a) Find a vector field such that $\mathbf{F}\cdot\mathbf{n}=x^4+y^4+z^4$ on the sphere S.

(b) Find $\iint_S(x^4+y^4+z^4)\,\mathrm{d}S$.

Solution.

General version of the Divergence Theorem, page 1144

Let's consider the region E that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2.

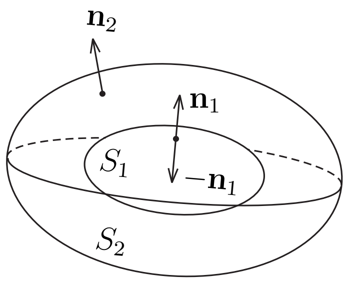

Figure 2: General version of the divergence theorem.

Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normals of S_1 and S_2. Then the boundary surface of E is $S=S_1\cup S_2$ and its normal $\mathbf{n}$ is given by $\mathbf{n}=-\mathbf{n}_1$ or $\mathbf{n}=\mathbf{n}_2$ on S_2. Applying the Divergence Theorem on S, we get

$$\begin{aligned}\iiint_E \operatorname{div}\mathbf{F}\mathrm{d}V &= \iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S\\ &= \iint_{S_1}\mathbf{F}\cdot(-\mathbf{n}_1)\,\mathrm{d}S + \iint_{S_2}\mathbf{F}\cdot\mathbf{n}_2\,\mathrm{d}S = -\iint_{S_1}\mathbf{F}\cdot\mathrm{d}\mathbf{S} + \iint_{S_2}\mathbf{F}\cdot\mathrm{d}\mathbf{S}.\end{aligned}$$

Example 5 (page 1144)**.** Show that the electric flux of $\mathbf{E}$ through any closed surface S_2 that encloses the origin is $\iint_{S_2} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 4\pi\varepsilon Q$.

Solution. We let S_1 be a small sphere with radius r_0 and center the origin. For the electric field $\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$, we have $\operatorname{div}\mathbf{E} = 0$. So

$$\iint_{S_2} \mathbf{E} \cdot \mathrm{d}\,\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathrm{d}\,\mathbf{S} + \iiint_E \operatorname{div}\mathbf{E}\,\mathrm{d}V = \iint_{S_1} \mathbf{E} \cdot \mathrm{d}\,\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n}\,\mathrm{d}S.$$

We can compute the surface integral over S_1 because S_1 is a sphere. The normal vector at $\mathbf{x}$ is $\frac{\mathbf{x}}{|\mathbf{x}|}$. Therefore

$$\mathbf{E} \cdot \mathbf{n} = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) = \frac{\varepsilon Q}{|\mathbf{x}|^4}\mathbf{x} \cdot \mathbf{x} = \frac{\varepsilon Q}{|\mathbf{x}|^2} = \frac{\varepsilon Q}{r_0^2}.$$

Thus we have

$$\iint_{S_2} \mathbf{E} \cdot \mathrm{d}\,\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathrm{d}\,\mathbf{S} = \frac{\varepsilon Q}{r_0^2} \iint_{S_1} 1\,\mathrm{d}S = \frac{\varepsilon Q}{r_0^2}\,\mathrm{Area}(S_1) = \frac{\varepsilon Q}{r_0^2} 4\pi r_0^2 = 4\pi\varepsilon Q.$$

Appendix, page 1141

Proof of the Divergence Theorem for simple solid regions. Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Then $\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$, so

$$\iiint_E \operatorname{div}\mathbf{F}\,\mathrm{d}V = \iiint_E \frac{\partial P}{\partial x}\,\mathrm{d}V + \iiint_E \frac{\partial Q}{\partial y}\,\mathrm{d}V + \iiint_E \frac{\partial R}{\partial z}\,\mathrm{d}V.$$

If $\mathbf{n}$ is the unit outward normal of S, then we have

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \mathbf{n}\,\mathrm{d}S \\ &= \iint_S P\,\mathbf{i} \cdot \mathbf{n}\,\mathrm{d}S + Q\,\mathbf{j} \cdot \mathbf{n}\,\mathrm{d}S + R\,\mathbf{k} \cdot \mathbf{n}\,\mathrm{d}S.\end{aligned}$$

So, to prove the Divergence Theorem, it suffices to prove the following equations:

$$\iint_S P\,\mathbf{i} \cdot \mathbf{n}\,\mathrm{d}S = \iiint_E \frac{\partial P}{\partial x}\,\mathrm{d}V, \qquad \iint_S Q\,\mathbf{j} \cdot \mathbf{n}\,\mathrm{d}S = \iiint_E \frac{\partial Q}{\partial y}\,\mathrm{d}V,$$
$$\iint_S R\,\mathbf{k} \cdot \mathbf{n}\,\mathrm{d}S = \iiint_E \frac{\partial R}{\partial z}\,\mathrm{d}V. \tag{1}$$

Here we only prove (1) and E is a type z region:

$$E = \{(x, y, z) | (x, y) \in D, z_1(x, y) \le z \le z_2(x, y)\},$$

By the Fundamental Theorem of Calculus, we have

$$\iiint_E \frac{\partial R}{\partial z}\,\mathrm{d}V = \iint_D \left(\int_{z=z_1(x,y)}^{z=z_2(x,y)} \frac{\partial R}{\partial z}(x,y,z)\,\mathrm{d}z \right) \mathrm{d}A$$
$$= \iint_D (R(x,y,z_2(x,y)) - R(x,y,z_1(x,y)))\,\mathrm{d}A.$$

On the other hand, the boundary surface S consists of three pieces: the bottom surface S_1, the top surface S_2, and a vertical surface S_3, which lies above the boundary curve of D.

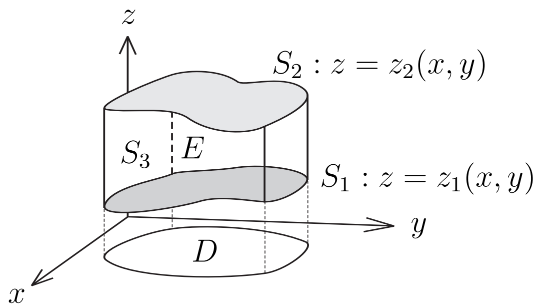

Figure 3: Proof of the Divergence Theorem.

We write

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,\mathrm{d}S + \iint_{S_2} R\,\mathbf{k}\cdot\mathbf{n}\,\mathrm{d}S + \iint_{S_3} R\,\mathbf{k}\cdot\mathbf{n}\,\mathrm{d}S.$$

On S_3, we have $\mathbf{k}\cdot\mathbf{n} = 0$, because $\mathbf{k}$ is vertical and $\mathbf{n}$ is horizontal, and so

$$\iint_{S_3} R\,\mathbf{k}\cdot\mathbf{n}\,\mathrm{d}S = \iint_{S_3} 0\,\mathrm{d}S = 0.$$

The equation of S_2 is $z = z_2(x,y), (x,y) \in D$, and the outward normal $\mathbf{n}$ points upward, so we have

$$\iint_{S_2} R\,\mathbf{k}\cdot\mathbf{n}\,\mathrm{d}S = \iint_D R(x,y,z_2(x,y))\,\mathrm{d}A.$$

On S_1 we have $z = z_1(x,y)$, but here the outward normal $\mathbf{n}$ points downward, so we multiply by -1:

$$\iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,\mathrm{d}S = -\iint_D R(x,y,z_1(x,y))\,\mathrm{d}A.$$

Therefore,

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,\mathrm{d}S = \iint_D (R(x,y,z_2(x,y)) - R(x,y,z_1(x,y)))\,\mathrm{d}A.$$

□