16.8 Stokes' Theorem, page 1134

Stokes' Theorem can be regarded as a generalization of Green's Theorem.

- Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve.
- Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve (space curve) of S.

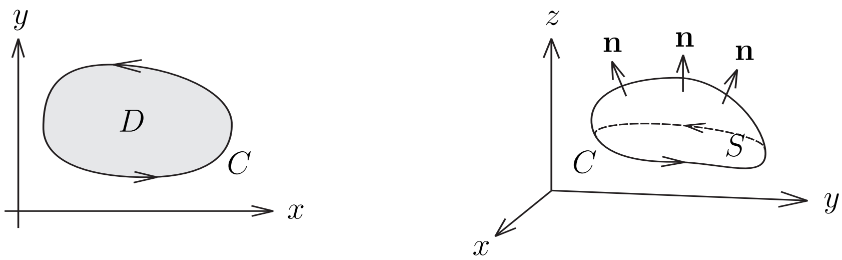

Figure 1: Green's Theorem (left) and Stokes' Theorem (right).

Figure 1 (right) shows an oriented surface with unit normal vector $\mathbf{n}$. The orientation of S induces the *positive orientation of the boundary curve* C (邊界曲線之正的定向) shown in the figure.

☐ 曲面的法向量 $\mathbf{n}$ 和邊界曲線 C 的定向滿足「右手定則」爲正的定向。

Stokes' Theorem (page 1134)**.** *Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region on $\mathbb{R}^3$ that contains S. Then*

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S}.$$

(a) Since

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \oint_C \mathbf{F}\cdot \mathbf{T}\,\mathrm{d}s \quad \text{and} \quad \iint_S \operatorname{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot \mathbf{n}\,\mathrm{d}S,$$

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral over S of the normal component of the curl of $\mathbf{F}$.

(b) Green's Theorem is the special case of Stokes' Theorem, where S is flat and lies in the xy-plane with upward orientation, and the unit normal is $\mathbf{k}$, so

$$\begin{aligned}\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A \\ &= \iint_S \operatorname{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot \mathbf{k}\,\mathrm{d}A.\end{aligned}$$

Example 1 (page 1136)**.** Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$. (Orient C to be counterclockwise when viewed from above.)

Solution.

Solution 2.

Example 2 (page 1137)**.** Compute the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside cylinder $x^2 + y^2 = 1$ and above the xy-plane.

Solution.

Solution 2.

Solution 3.

☐ 若兩定向曲面具有相同的邊界, 則面積分 $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ 的值相同(微積分基本定理)。

Example 3. Suppose that S consists of the part of cylinder $x^2 + y^2 = 1, 0 \leq z \leq 1$ and the lid $x^2 + y^2 \leq 1, z = 1$. Let $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + x^2\,\mathbf{k}$. Evaluate $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S$, where S is oriented outward viewed from the origin.

Solution.

Solution 2.

Example 4. Suppose that C is the circle that is the intersection of the plane passing through the origin and the sphere $x^2 + y^2 + z^2 = 4$. Let $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$. Find the equation of the plane such that the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ attains the maximum.

Solution.

Appendix, page 1135

Proof of a special case of Stokes' Theorem. We assume that the equation of S is $z = z(x, y)$, $(x, y) \in D$, where $z(x, y)$ has continuous second order partial derivatives and D is a simple plane region whose boundary curve C_1 corresponds to C. If the orientation of S is upward, then the positive orientation of C corresponds to the positive orientation of C_1. So we have $\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + z(x, y)\,\mathbf{k}$ and

$$\mathbf{r}_x(x, y) = 1\,\mathbf{i} + 0\,\mathbf{j} + z_x\,\mathbf{k}, \qquad \mathbf{r}_y(x, y) = 0\,\mathbf{i} + 1\,\mathbf{j} + z_y\,\mathbf{k},$$
$$\mathbf{r}_x \times \mathbf{r}_y(x, y) = -z_x\,\mathbf{i} - z_y\,\mathbf{j} + 1\,\mathbf{k}.$$

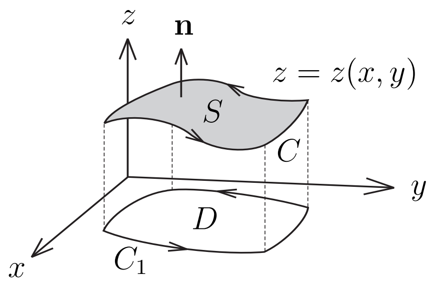

Figure 2: Proof of Stokes' Theorem.

Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. We first compute

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_D (-(R_y - Q_z)z_x - (P_z - R_x)z_y + (Q_x - P_y))\,\mathrm{d}A,$$

where the partial derivatives of P, Q, and R are evaluated at $(x, y, z(x, y))$. On the other hand, if C_1 is given by $\mathbf{r}_1(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, where t is from a to b, then C is given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(x(t), y(t))\,\mathbf{k}$, where t is from a to b. So

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b (Px'(t) + Qy'(t) + Rz'(t))\,\mathrm{d}t$$
$$= \int_a^b (Px'(t) + Qy'(t) + R(z_x x'(t) + z_y y'(t)))\,\mathrm{d}t$$
$$= \int_a^b ((P + Rz_x)x'(t) + (Q + Rz_y)y'(t))\,\mathrm{d}t = \int_{C_1} (P + Rz_x)\,\mathrm{d}x + (Q + Rz_y)\,\mathrm{d}y$$
$$= \iint_D \left(\frac{\partial}{\partial x}(Q + Rz_y) - \frac{\partial}{\partial y}(P + Rz_x)\right)\mathrm{d}A. \qquad \text{(by Green's Theorem)}$$

Using the Chain Rule carefully, that is, P, Q, and R are functions of x, y, and z and z itself a function of x and y, we will get

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \begin{pmatrix} (Q_x + Q_z z_x + R_x z_y + R_z z_x z_y + R z_{xy}) \\ -(P_y + P_z z_y + R_y z_x + R_z z_y z_x + R z_{yx}) \end{pmatrix} \mathrm{d}A$$
$$= \iint_D (-(R_y - Q_z)\,z_x - (P_z - R_x)\,z_y + (Q_x - P_y))\,\mathrm{d}A.$$

□