16.7 Surface Integrals, page 1122

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length.

Suppose f is a function of three variables whose domain includes a surface S. First, we will define the surface integral of f over S.

Surface integrals of functions (第一類曲面積分), page 1123

Suppose that a surface S has a vector equation

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}, \qquad (u,v) \in D.$$

(1) We first assume that the parameter domain D is a rectangle and we divide it into subrectangles R_{ij} with dimensions Δu and Δv. The surface S is divided into corresponding patches S_{ij}.

(2) We evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij}.

(3) We form the Riemann sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$.

(4) Taking the limit as the number of patches increasing and define the *surface integral of f over the surface S* (第一類曲面積分) as

$$\iint_S f(x,y,z)\,\mathrm{d}S = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij} = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}A.$$

☐ 和第一類曲線積分比較: $\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,\mathrm{d}t$.

Example 1 (page 1123)**.** Compute the surface integral $\iint_S x^2\,\mathrm{d}S$, where S is the unit sphere $x^2+y^2+z^2=1$.

Solution.

Surface integrals have applications. For example, if a thin sheet has the shape of a surface S and density $\rho(x, y, z)$, then the total *mass* (質量) of the sheet is

$$m = \iint_S \rho(x, y, z)\, \mathrm{d}S.$$

The *center of mass* (質心) of the sheet is

$$\bar{x} = \frac{1}{m} \iint_S x\rho(x, y, z)\, \mathrm{d}S \quad \bar{y} = \frac{1}{m} \iint_S y\rho(x, y, z)\, \mathrm{d}S \quad \bar{z} = \frac{1}{m} \iint_S z\rho(x, y, z)\, \mathrm{d}S.$$

Graph, page 1124

Example 2 (page 1124)**.** Any surface S with equation $z = z(x, y)$ can be regarded as a parameter surface with parametric equations

$$x = x \qquad y = y \qquad z = z(x, y).$$

So we have

Similar formulas apply when we project S onto the yz-plane or xz-plane.

(a) If S is a surface with equation $y = y(x, z)$ and D is its projection onto the xz-plane, then

$$\iint_S f(x, y, z)\, \mathrm{d}S = \iint_D f(x, y(x, z), z) \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, \mathrm{d}A.$$

(b) If S is a surface with equation $x = x(y, z)$ and D is its projection onto the xz-plane, then

$$\iint_S f(x, y, z)\, \mathrm{d}S = \iint_D f(x(y, z), y, z) \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, \mathrm{d}A.$$

If S is a piecewise smooth surface, that is, a finite union of smooth surfaces $S_1, S_2, \ldots, S_n$ that intersect only along their boundaries, then the surface integral of f over S is defined by

$$\iint_S f(x, y, z)\, \mathrm{d}S = \iint_{S_1} f(x, y, z)\, \mathrm{d}S + \cdots + \iint_{S_n} f(x, y, z)\, \mathrm{d}S.$$

Example 3 (page 1125)**.** Evaluate $\iint_S z\,\mathrm{d}S$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leq 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2.

Solution.

Oriented surfaces, page 1127

To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip. The Möbius strip is a “one-sided surface.”

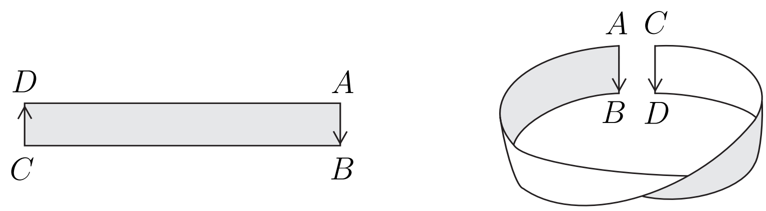

Figure 1: Möbius strip.

From now on we consider only orientable (two-sided) surfaces. We start with a surface S that has a tangent plane at every point (x, y, z) on S except at any boundary point. There are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z).

Definition 4 (page 1127)**.** If it is possible to choose a unit normal vector $\mathbf{n}$ at every such point (x, y, z) so that $\mathbf{n}$ varies continuously over S, then S is called an *oriented surface* (可定向的曲面) and the given choice of $\mathbf{n}$ provides S with an *orientation* (定向). There are two possible orientations for any orientable surface.

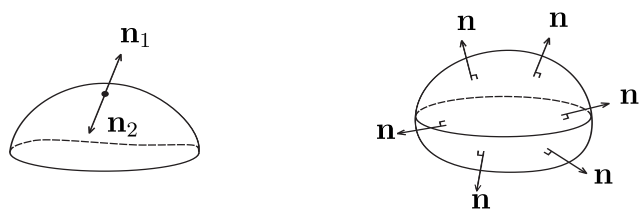

Figure 2: Oriented surfaces.

(a) If S is a smooth orientable surface given by $\mathbf{r}(u, v)$, then the orientation of the unit normal vector is

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

and opposite orientation is given by $-\mathbf{n}$.

(b) For a surface S given as the graph of $z = z(x, y)$, we can get the unit normal vector

$$\mathbf{n} = \frac{-z_x\,\mathbf{i} - z_y\,\mathbf{j} + \mathbf{k}}{\sqrt{1 + z_x^2 + z_y^2}}.$$

The unit normal vector gives the *upward orientation* of the surface.

(c) For a closed surface S (the boundary of a solid region E), the convention is that the positive orientation is the normal vectors point outward from E, and inward-pointing normals give the negative orientation.

Surface integrals of vector fields (第二類曲面積分), page 1128

Suppose that S is an orientated surface with unit normal vector $\mathbf{n}$, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing through S. Then the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$. If we divide S into small patches S_{ij}, then S_{ij} is nearly planar and so we can approximate the mass of fluid per unit time crossing S_{ij} in the direction of the normal $\mathbf{n}$ by the quantity $(\rho\mathbf{v}\cdot\mathbf{n})A(S_{ij})$, where $\rho, \mathbf{v}$, and $\mathbf{n}$ are evaluated at some point on S_{ij}. By summing these quantities and taking the limit, we get the surface integral of the function $\rho\mathbf{v}\cdot\mathbf{n}$ over S:

$$\iint_S \rho\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = \iint_S \rho(x, y, z)\mathbf{v}(x, y, z)\cdot\mathbf{n}(x, y, z)\,\mathrm{d}S.$$

If we write $\mathbf{F} = \rho\mathbf{v}$, then $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and the integral becomes $\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$. Such surface integral is called the *surface integral* (第二類曲面積分), or *flux integral* (通量積分) of $\mathbf{F}$ over S.

Definition 5 (page 1129)**.** If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the *surface integral of* $\mathbf{F}$ *over* S is

$$\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S.$$

This integral is also called the *flux* (通量) of $\mathbf{F}$ across S.

The surface integral of a vector field over S is equal to the surface integral of its normal component over S.

(a) If S is given by a vector function $\mathbf{r}(u, v)$, then we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \iint_D \left(\mathbf{F}(\mathbf{r}(u, v))\cdot\pm\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}\right)|\mathbf{r}_u\times\mathbf{r}_v|\,\mathrm{d}A\\ &= \iint_D \mathbf{F}(\mathbf{r}(u, v))\cdot(\pm\mathbf{r}_u\times\mathbf{r}_v)\,\mathrm{d}A.\end{aligned}$$

(b) If a surface S is given by a graph $z = z(x, y)$, then we can use vector function $\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + g(x, y)\,\mathbf{k}$ and get

$$\begin{aligned}\iint_S \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k})\cdot(\mp z_x\,\mathbf{i}\mp z_y\,\mathbf{j}\pm\mathbf{k})\,\,\mathrm{d}A\\ &= \iint_D (\mp Pg_x \mp Qg_y \pm R)\,\,\mathrm{d}A.\end{aligned}$$

(c) Is a surface S is a level surface $g(x, y, z) = 0$, then $\mathbf{n} = \pm\frac{\nabla g}{|\nabla g|}$, and

$$\iint_S \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} = \iint_D \mathbf{F}\cdot\left(\pm\frac{\nabla g}{|\nabla g|}\right)\frac{|\nabla g|}{|\nabla g\cdot\mathbf{p}|}\,\mathrm{d}A = \iint_D \mathbf{F}\cdot\left(\pm\frac{\nabla g}{|\nabla g\cdot\mathbf{p}|}\right)\,\mathrm{d}A,$$

where $\mathbf{p}$ is the unit normal vector to the plane region.

Example 6 (page 1130)**.** Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution.

Example 7 (page 1130)**.** Evaluate $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, where $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution.

Appendix, page 1131

Definition 8 (page 1131)**.** If $\mathbf{E}$ is an electric field, then the surface integral $\iint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S}$ is called the *electric flux of E* (電通量) through the surface S.

One of the important laws of electrostatics is *Gauss's Law* (高斯定律), which says that the net charge enclosed by a closed surface S is

$$Q = \varepsilon_0 \iint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S},$$

where ε_0 is a constant (called the permittivity of free space 真空電容率) that depends on the units used. Therefore, if the vector field $\mathbf{F} = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ represents an electric field, we can conclude that the charge enclosed by S is $Q = \frac{4}{3}\pi\varepsilon_0$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the *heat flow* is defined as the vector field

$$\mathbf{F} = -K\nabla u,$$

where K is an experimentally determined constant called the *conductivity* of the substance. The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = -K \iint_S \nabla u \cdot \mathrm{d}\mathbf{S}.$$

Example 9 (page 1132)**.** The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius R with center at the center of the ball.

Solution.