16.6 Parametric Surfaces and Their Areas, page 1111

Here we use vector functions to describe more general surfaces, called *parametric surfaces*, and compute their areas.

Parametric surfaces (參數曲面), page 1111

We can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v:

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k},$$

which is a vector-valued function defined on a region D in the uv-plane. So $x(u, v)$, $y(u, v)$, and $z(u, v)$, called the component functions (分量函數) of $\mathbf{r}(u, v)$.

Definition 1 (page 1111)**.** The set of all points (x, y, z) in $\mathbb{R}^3$ such that

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \tag{1}$$

where (u, v) varies throughout D, is called *parametric surfaces* S (參數曲面). Equations (1) are called *parametric equations* of S (參數方程).

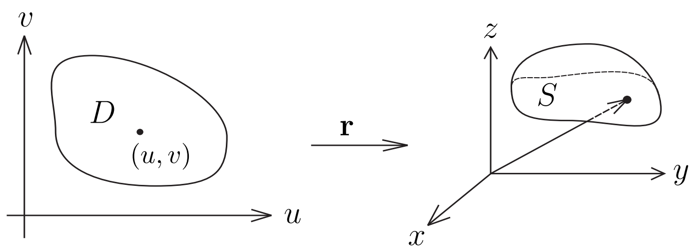

Figure 1: Parametric surface.

Example 2 (page 1113)**.** Find a vector function of the following surfaces.

(a) Plane passes through P_0 with position vector $\mathbf{r}_0$ and contains vectors $\mathbf{a}$ and $\mathbf{b}$.

(b) Sphere $x^2 + y^2 + z^2 = r_0^2$.

(c) Cylinder $x^2 + z^2 = r_0^2$.

Solution.

Definition 3 (page 1112)**.** If we keep u constant by putting $u = u_0$, we get a curve C_v given by $\mathbf{r}(u_0, v)$. If we keep v constant by putting $v = v_0$, we get a curve C_u given by $\mathbf{r}(u, v_0)$. These curves are called *grid curves* (網格曲線).

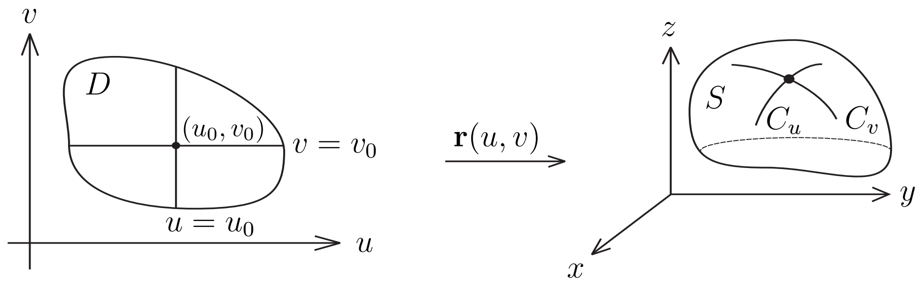

Figure 2: Grid curves.

Example 4 (page 1114)**.** Find a vector function of the following surfaces. (a) Elliptic paraboloid $z = x^2 + 2y^2$. (b) The top half of the cone $z = 2\sqrt{x^2 + y^2}$.

Solution.

Surfaces of revolution (旋轉曲面), page 1115

Consider the surface of revolution S obtained by rotating the curve $y = f(x), a \leq x \leq b$, about the x-axis, where $f(x) \geq 0$. Let θ be the angle of rotation. If (x, y, z) is a point on S, then

$$x = x, \quad y = f(x)\cos\theta, \quad z = f(x)\sin\theta. \tag{2}$$

Therefore we take x and θ as parameters and regard equation (2) as parameter equations of S. The parameter domain is given by $a \leq x \leq b, 0 \leq \theta \leq 2\pi$.

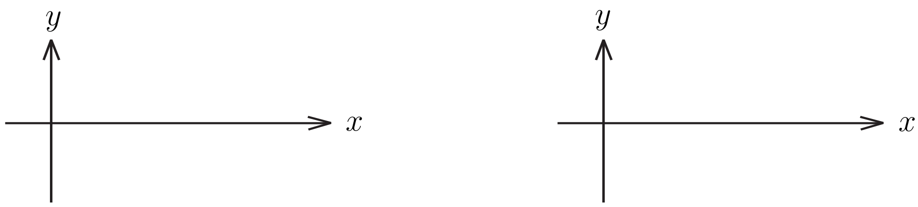

Figure 3: Left: Surface of revolution; Right: **Example 5**.

Example 5 (page 1115)**.** Find parametric equations for the surface generated by rotating the curve $y = \sin x, 0 \leq x \leq 2\pi$, about x-axis.

Solution.

Tangent planes, page 1115

We now find the tangent plane to a parametric surface S traced out by

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$.

If we keep v constant by putting $v = v_0$, then $\mathbf{r}(u, v_0)$ becomes a vector function of the single parameter u and defines a grid curve C_u lying on S. The tangent vector to C_u at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to u:

$$\mathbf{r}_u(u_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\mathbf{k}.$$

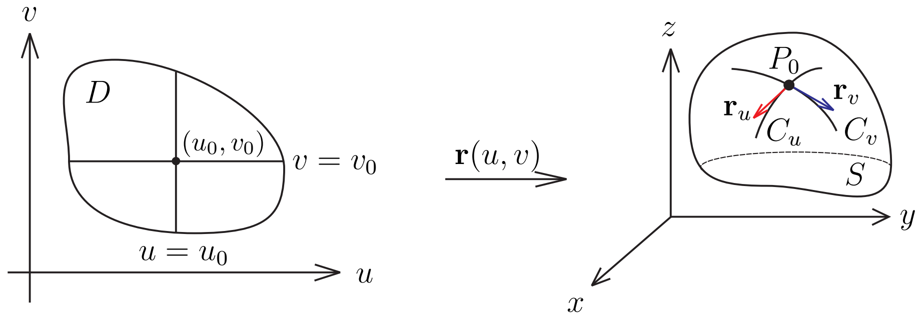

Figure 4: Tangent plane is spanned by $\mathbf{r}_u$ and $\mathbf{r}_v$.

If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_v lying on S. The tangent vector to C_v at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

$$\mathbf{r}_v(u_0, v_0) = \frac{\partial x}{\partial v}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\mathbf{k}.$$

If $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface S is called *smooth* (光滑曲面), and it has no corners. For a smooth surface, the *tangent plane* (切平面) is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

Surface area, page 1116

Definition 6 (page 1117)**.** If a smooth parametric surface S is given by

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}, \qquad (u, v) \in D,$$

and S is covered just once as (u, v) ranges throughout the parametric domain D, then the *surface area* of S (曲面面積) is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}A,$$

where $\mathbf{r}_u = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k}$ and $\mathbf{r}_v = \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j} + \frac{\partial z}{\partial v}\,\mathbf{k}$.

Example 7 (page 1117)**.** Find the surface area of a sphere of radius R.

Solution.

Surface area of the graph of a function, page 1118

Example 8 (page 1118)**.** If a surface S is a graph of $z = f(x, y)$, then the parameter equation of S is

$$\mathbf{r}(x, y) =$$

We compute

Example 9 (page 1119)**.** For surfaces of revolution, prove that the surface area formula is consistent with the surface area formula from single-variable calculus.

Solution.