16.5 Curl and Divergence, page 1103

Curl (旋度), page 1103

Definition 1 (page 1103). If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of $P(x, y, z), Q(x, y, z)$, and $R(x, y, z)$ all exist, then the *curl of* **F** (旋度) is the vector field on $\mathbb{R}^3$ defined by

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

☐ 有些書籍或文獻會用 $\operatorname{rot}\mathbf{F}$ 表示旋度。

We introduce the vector differential operator ∇ as

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

(a) When it operates on a scalar function, we get the gradient of f:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

(b) If we think of ∇ as a vector with components $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, then $\operatorname{curl}\mathbf{F}$ is the formal cross product of ∇ with the vector field **F** as follows:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \operatorname{curl}\mathbf{F}.$$

Theorem 2 (page 1104). *If $f(x, y, z)$ is a function that has continuous second order partial derivatives, then* $\operatorname{curl}(\nabla f) = \mathbf{0}$.

Proof. By Clairaut's Theorem, we have

$$\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$=$$

☐

Corollary 3 (page 1104). *If* **F** *is a conservative vector field, then* $\operatorname{curl}\mathbf{F} = 0$.

Theorem 4 (page 1105). *If* **F** *is a vector field defined on all of* $\mathbb{R}^3$ (*more generally if the domain is simply connected*) *whose component functions have continuous partial derivatives and* $\operatorname{curl}\mathbf{F} = \mathbf{0}$, *then* **F** *is a conservative vector field.*

Divergence (散度), page 1106

Definition 5 (page 1106)**.** If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, then the *divergence of* $\mathbf{F}$ (散度) is the function of three variables defined by

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

(a) The divergence of $\mathbf{F}$ is the dot product of ∇ and $\mathbf{F}$: $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}$.

(b) $\operatorname{curl}\mathbf{F}$ is a vector field but $\operatorname{div}\mathbf{F}$ is a scalar field.

Theorem 6 (page 1106)**.** *If* $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ *is a vector field on* $\mathbb{R}^3$ *and* $P, Q,$ *and* R *have continuous second order partial derivatives, then* $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$.

Proof. Using the definition of divergence and curl, we have

$$\begin{aligned}\operatorname{div}(\operatorname{curl}\mathbf{F}) &= \nabla \cdot (\nabla \times \mathbf{F}) \\ &= \\ &= \end{aligned}$$

because the terms cancel in pairs by Clairaut's Theorem. □

Example 7 (page 1107)**.** Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \operatorname{curl}\mathbf{G}$.

Solution.

Definition 8 (page 1107)**.**

(a) Define the operator $\nabla^2 = \nabla \cdot \nabla$.

(b) The *Laplace operator* of a function f is defined as:

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

(c) We can apply the Laplace operator to a vector field $\mathbf{F}$:

$$\nabla^2\mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}.$$

Vector forms of Green's Theorem, page 1108

The curl and divergence operators allow us to rewrite Green's Theorem in versions that will be useful in our later work. We suppose that the plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem. Then we consider the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$. Its line integral is

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y,$$

and, regarding $\mathbf{F}$ as a vector field on $\mathbb{R}^3$ with third component 0, we have

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.$$

So we can rewrite Green's Theorem in the vector form

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\,\mathrm{d}A.$$

Now we will derive a similar formula involving the *normal component* of $\mathbf{F}$. If C is given by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, a \le t \le b$, then the unit tangent vector $\mathbf{T}(t)$ and unit outward normal vector $\mathbf{n}(t)$ to C are given by

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}, \quad \text{and} \quad \mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}.$$

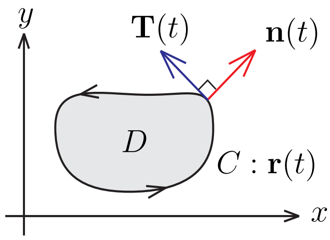

Figure 1: Unit tangent vector $\mathbf{T}(t)$ and unit normal vector $\mathbf{n}(t)$ of the curve $C : \mathbf{r}(t)$.

So we have

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s &= \int_a^b (\mathbf{F} \cdot \mathbf{n}) |\mathbf{r}'(t)|\,\mathrm{d}t \\ &= \int_a^b \left(\frac{P(x(t), y(t)) y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t)) x'(t)}{|\mathbf{r}'(t)|} \right) |\mathbf{r}'(t)|\,\mathrm{d}t \\ &= \int_a^b P(x(t), y(t)) y'(t)\,\mathrm{d}t - Q(x(t), y(t)) x'(t)\,\mathrm{d}t \\ &= \int_a^b P\,\mathrm{d}y - Q\,\mathrm{d}x = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \mathrm{d}A. \end{aligned}$$

Appendix

Physics meaning of curl and divergence, page 1106–1107

The reason for the name "curl" occurs when $\mathbf{F}$ represents the velocity field in fluid flow. Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of $\operatorname{curl}\mathbf{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis.

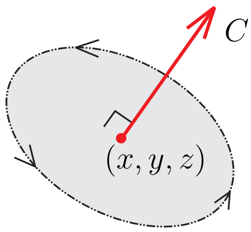

Figure 2: Curl of a vector field $\mathbf{F}$.

- If $\operatorname{curl}\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and $\mathbf{F}$ is called *irrotational* (無旋) at P.
- If $\operatorname{curl}\mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis.

The reason for the name "divergence" can be understood in the context of fluid flow. If $\mathbf{F}(x, y, z)$ is the velocity of a fluid or gas, then $\operatorname{div}\mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid flowing from the point (x, y, z) per unit volume. In other words, $\operatorname{div}\mathbf{F}$ measures the tendency of the fluid to diverge from the point (x, y, z).

- If $\operatorname{div}\mathbf{F} = 0$, then $\mathbf{F}$ is said to be *incompressible* (無壓縮的).

Exercise (page 1110)**.** Let f be a scalar field and $\mathbf{F}$ a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field. (a) $\operatorname{curl} f$. (b) $\operatorname{grad} f$. (c) $\operatorname{div}\mathbf{F}$. (d) $\operatorname{curl}(\operatorname{grad} f)$. (e) $\operatorname{grad}\mathbf{F}$ (f) $\operatorname{grad}(\operatorname{div}\mathbf{F})$ (g) $\operatorname{div}(\operatorname{grad} f)$ (h) $\operatorname{grad}(\operatorname{div} f)$ (i) $\operatorname{curl}(\operatorname{curl}\mathbf{F})$ (j) $\operatorname{div}(\operatorname{div}\mathbf{F})$ (k) $(\operatorname{grad} f) \times (\operatorname{div}\mathbf{F})$ (l) $\operatorname{div}(\operatorname{curl}(\operatorname{grad} f))$.