16.4 Green's Theorem, page 1096

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

Definition 1 (page 1096)**.** We say a simple closed curve C is *positive orientation* (正的定向) if the curve is traverses counterclockwise.

If C is given by the vector function $\mathbf{r}(t), a \leq t \leq b$, then the region D is always on the left at the point $\mathbf{r}(t)$ traverses C.

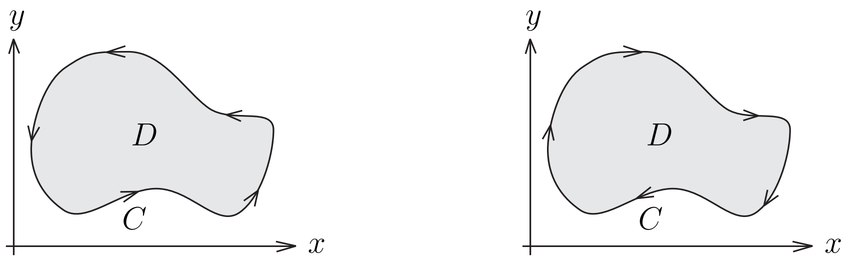

Figure 1: Positive orientation (left) and negative orientation (right).

Green's Theorem (page 1096)**.** *Let C be a positive oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region that contains D, then*

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A.$$

Remark 2. Sometimes we use the following notations

$$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y, \qquad \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y, \quad \text{or} \quad \int_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y$$

to indicate that the line integral is calculated in the positive orientation.

Example 3 (page 1098)**.** Evaluate $\int_C x^4\,\mathrm{d}x + xy\,\mathrm{d}y$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, form $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

Solution.

Example 4 (page 1098)**.** Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2 + y^2 = 9$.

Solution.

Example 5 (page 1098)**.** If $P(x,y) = Q(x,y) = 0$ on a simple closed curve C, and $P(x,y), Q(x,y)$ satisfy the hypotheses of Green's Theorem, then

no matter what values P and Q assume in the region D.

Example 6 (page 1099)**.** If we take $(P,Q) = (0,x)$, $(P,Q) = (-y,0)$, and $(P,Q) = (-\frac{1}{2}y, \frac{1}{2}x)$, then Green's Theorem gives

Example 7 (page 1102)**.**

(a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), then

$$\int_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy = \frac{1}{2}(x_1y_2 - x_2y_1).$$

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots,$ (x_n, y_n), then the area of the polygon is

$$A = \frac{1}{2}((x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - y_{n-1}x_n) + (x_ny_1 - y_nx_1)).$$

Extended Versions of Green's Theorem, page 1099

Green's Theorem can be extended to apply to regions with holes (genus), that is, regions that are not simply connected.

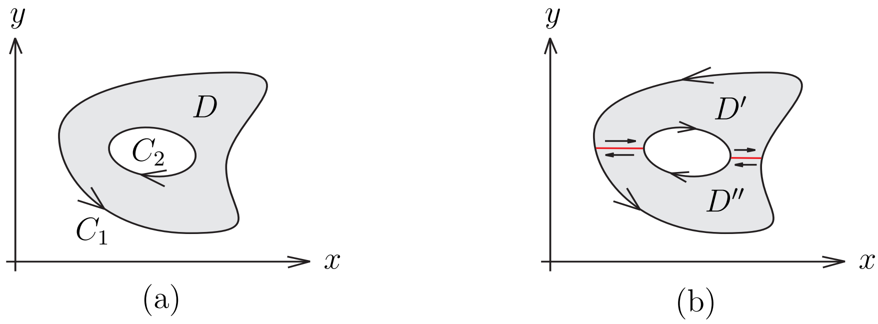

Figure 2: Region D is not simply connected.

See Figure 2 (a). Observe that the boundary C of the region D consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus the positive direction is *counterclockwise* for the outer curve C_1 but *clockwise* for the inner curve C_2.

If we divide D into two region D' and D'' by means of the lines shown in Figure 2 (b), then we applying Green's Theorem to each of D' and D'' to get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A$$
$$= \int_{\partial D'} P\,\mathrm{d}x + Q\,\mathrm{d}y + \int_{\partial D''} P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \int_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y + \int_{C_2} P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

Example 8 (page 1100). Let $\mathbf{F}(x, y) = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$.

(a) Show that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for every simple closed curve that does not encloses the origin.

(b) Show that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

Solution.

Appendix, page 1097

Proof of Green's Theorem in which D is a simple region. It suffices to show that

$$\int_C P(x,y)\,\mathrm{d}x = -\iint_D \frac{\partial P}{\partial y}\,\mathrm{d}A \quad \text{and} \quad \int_C Q(x,y)\,\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x}\,\mathrm{d}A.$$

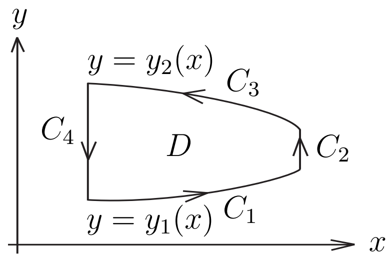

Figure 3: Simple Region D.

We express D as a type I region $D = \{(x,y) |\, a \le x \le b, y_1(x) \le y \le y_2(x)\}$, where $y_1(x)$ and $y_2(x)$ are continuous functions. By the Fundamental Theorem of Calculus, we have

$$-\iint_D \frac{\partial P}{\partial y}\,\mathrm{d}A = -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y}(x,y)\,\mathrm{d}y\,\mathrm{d}x = -\int_a^b (P(x,y_2(x)) - P(x,y_1(x)))\,\mathrm{d}x.$$

On the other hand, we know $C = C_1 \cup C_2 \cup C_3 \cup C_4$. On C_1, we write the vector function $\mathbf{r}_1(t) = t\,\mathbf{i} + y_1(t)\,\mathbf{j}$, and t from a to b. So

$$\int_{C_1} P(x,y)\,\mathrm{d}x = \int_a^b P(x,y_1(x))\,\mathrm{d}x.$$

On C_3, we use the vector function $\mathbf{r}_3(t) = t\,\mathbf{i} + y_2(t)\,\mathbf{j}$, t from b to a. Therefore

$$\int_{C_3} P(x,y)\,\mathrm{d}x = \int_b^a P(t,y_2(t))\,\mathrm{d}t = -\int_a^b P(x,y_2(x))\,\mathrm{d}x.$$

On C_2 or C_4, x is constant, so $\mathrm{d}x = 0$ and hence

$$\int_{C_2} P(x,y)\,\mathrm{d}x = 0 = \int_{C_4} P(x,y)\,\mathrm{d}x.$$

Hence

$$\begin{aligned}\int_C P(x,y)\,\mathrm{d}x &= \int_{C_1} P(x,y)\,\mathrm{d}x + \int_{C_2} P(x,y)\,\mathrm{d}x + \int_{C_3} P(x,y)\,\mathrm{d}x + \int_{C_4} P(x,y)\,\mathrm{d}x \\ &= \int_a^b P(x,y_1(x))\,\mathrm{d}x - \int_a^b P(x,y_2(x))\,\mathrm{d}x = -\iint_D \frac{\partial P}{\partial y}\,\mathrm{d}A.\end{aligned}$$

Equality $\int_C Q(x,y)\,\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x}\,\mathrm{d}A$ can be proved similarly. □

Proof of Theorem 10 in section 16.3. If C is any simple closed path in D and R is the region that encloses, then Green's Theorem gives

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_C P \,\mathrm{d}x + Q \,\mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A = \iint_R 0 \,\mathrm{d}A = 0.$$

A curve that is not simple crossed itself at one or more points and can be broken up into a number of simple curve. We have shown that the line integral of $\mathbf{F}$ around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for any closed curve C. Therefore $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path in D, and $\mathbf{F}$ is a conservative vector field. ☐

Remark 9. In differential geometry, we define the "wedge product" or "exterior operator" on vectors or differential forms. Given two differential forms $\mathrm{d}x, \mathrm{d}y$, their wedge product $\mathrm{d}x \wedge \mathrm{d}y$ means the positive oriented area element, so we have

$$\mathrm{d}A = \mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x, \quad \text{and} \quad \mathrm{d}x \wedge \mathrm{d}x = 0, \quad \text{and} \quad \mathrm{d}(\mathrm{d}x) = 0.$$

Green's Theorem can be regarded as the relationship between the integral, differential forms, and wedge product:

$$\begin{aligned}
&\int_C P \,\mathrm{d}x + Q \,\mathrm{d}y = \iint_D \mathrm{d}(P \,\mathrm{d}x + Q \,\mathrm{d}y) \\
&= \iint_D \left(\frac{\partial P}{\partial x} \mathrm{d}x + \frac{\partial P}{\partial y} \mathrm{d}y \right) \wedge \mathrm{d}x + P \,\mathrm{d}(\mathrm{d}x) + \left(\frac{\partial Q}{\partial x} \mathrm{d}x + \frac{\partial Q}{\partial y} \mathrm{d}y \right) \wedge \mathrm{d}y + Q \,\mathrm{d}(\mathrm{d}y) \\
&= \iint_D \frac{\partial P}{\partial y} \mathrm{d}y \wedge \mathrm{d}x + \int_D \frac{\partial Q}{\partial x} \mathrm{d}x \wedge \mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A.
\end{aligned}$$