16.3 The Fundamental Theorem for Line Integrals, page 1087

Recall that Part 2 of the Fundamental Theorem of Calculus is

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a), \tag{1}$$

where $F'(x)$ is continuous on $[a,b]$. We also called equation (1) the Net Change Theorem: The integral of a rate of change is the net change.

Here we will introduce the Fundamental Theorem for line integrals, where we think of the gradient vector ∇f of a function f as a sort of derivative of f.

Theorem 1 (page 1087)**.** *Let C be a smooth curve given by the vector function $\mathbf{r}(t), a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

☐ 若向量場來自於函數的梯度 (保守向量場), 則線積分的值爲兩端點函數值的差。

☐ 第二類曲線積分的路徑有方向性, 即 $\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = -\int_{-C} \nabla f \cdot \mathrm{d}\mathbf{r}$。

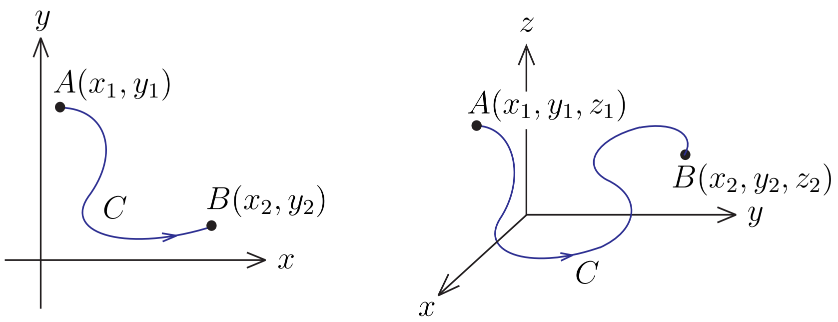

Figure 1: The fundamental theorem for line integrals.

Proof. By the Chain Rule and the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\int_C \nabla f \cdot \mathrm{d}\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_a^b \left(\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t))\,\mathrm{d}t = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).\end{aligned}$$

☐

Theorem 1 is also true for piecewise smooth curves. This can be seen by subdividing C into a finite number of smooth curves and adding the resulting integrals.

Example 2 (page 1088)**.** Find the work done by the gravitational field $\mathbf{F}(\mathbf{x}) = -\frac{GMm}{|\mathbf{x}|^3}\mathbf{x}$ in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise smooth curve C.

Solution.

Independence of Path, page 1088

Suppose that C_1 and C_2 are two piecewise smooth curves, which are also called *paths* (路徑) with have the same initial point A and terminal point B.

Definition 3 (page 1088)**.** Suppose that $\mathbf{F}$ is a continuous vector field with domain D. We say the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is *independent of path* (積分和路徑選取無關) if

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

for *any* two path C_1 and C_2 in D with the same initial and terminal points.

- In general vector field $\mathbf{F}$, $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. (See 16.2, **Example 5**).
- For *conservative vector field* $\mathbf{F} = \nabla f$, the Fundamental Theorem for line integrals tells us $\int_{C_1} \nabla f \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.
- The following discussion will say that the *only* vector fields that are independent of path are conservative vector fields.

☐ 保守向量場 ⇒ 第二類曲線積分與路徑選取無關。

Definition 4 (page 1089)**.** A curve is called *closed* (封閉曲線) if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$.

Figure 2: Closed curve (left) and non-closed curve (right).

Theorem 5 (page 1089)**.** *The line integral* $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ *is independent of path in* D *if and only if* $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ *for every closed path* C *in* D*.*

Proof. ($\Rightarrow$) We choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A. Then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1 \cup C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0.$$

($\Leftarrow$) For any paths C_1 and C_2 from A to B in D, we define C to be the curve consisting of C_1 followed by $-C_2$. Then we get

$$0 = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

and hence $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. □

□ 第二類曲線積分與路徑無關 $\Leftrightarrow$ 封閉曲線上的第二類曲線積分爲零。

□ 在保守向量場 (例如重力場) 將一物沿封閉曲線做功爲零。

Definition 6 (page 1089)**.**

(a) A domain D is *open* (開集合) if for every point P in D, there is a disk with center P that lies entirely in D. (D doesn't contain any of its boundary points.)

(b) A domain D is *path connected* (路徑連通) if any two points in D can be joined by a path that lies in D.

Figure 3: Open set; non-open set; path connected region; non-path connected region.

Theorem 7 (page 1089)**.** *Suppose* $\mathbf{F}$ *is a vector field that is continuous on an open, path connected region* D*. If* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *is independent of path in* D*, then* $\mathbf{F}$ *is a conservative vector field on* D*; that is, there exists a function* f *such that* $\nabla f = \mathbf{F}$*.*

Proof. Let $A(a,b)$ be a fixed point in D. We construct the potential function f by

$$f(x,y) = \int_{(a,b)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r}$$

for $(x,y) \in D$. Since $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path, the function is well-defined. Now we will show that $\nabla f = \mathbf{F}$:

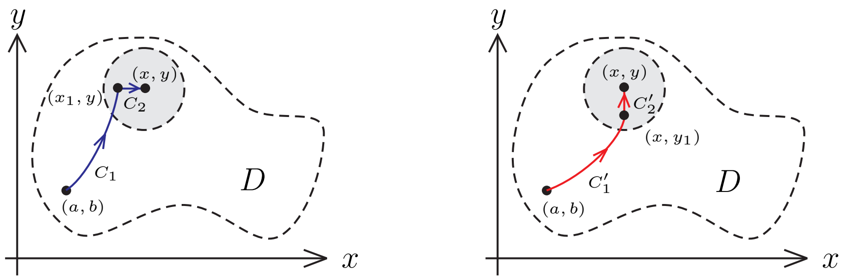

Figure 4: Choose suitable paths to prove $\nabla f = \mathbf{F}$.

Since D is open, there exists a disk contained in D with center (x,y). Choose any point (x_1,y) in the disk with $x_1 < x$ and let C consists of any path C_1 from (a,b) to (x_1,y) followed by the horizontal line segment C_2 from (x_1,y) to (x,y). Then

$$f(x,y) = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{(a,b)}^{(x_1,y)} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}.$$

Notice that the first of these integrals does *not* depend on x, so

$$\frac{\partial}{\partial x} f(x,y) = 0 + \frac{\partial}{\partial x}\int_{C_2} \mathbf{F}\cdot d\mathbf{r}. \tag{2}$$

Consider $C_2 : \mathbf{r}(t) = t\,\mathbf{i} + y\,\mathbf{j}$, where t from x_1 to x, then $\mathbf{r}'(t) = 1\,\mathbf{i} + 0\,\mathbf{j}$ and $\mathbf{F}(t) = P(t,y)\,\mathbf{i} + Q(t,y)\,\mathbf{j}$. Thus (2) gives

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \frac{\partial}{\partial x}\int_{x_1}^{x} P(t,y)\,dt = P(x,y).$$

Similarly, using a vertical line segment, consider $C = C_1' \cup C_2'$, $C_2' : \mathbf{r}(t) = x\,\mathbf{i} + t\,\mathbf{j}$, where t from y_1 to y, then $\mathbf{r}'(t) = 0\,\mathbf{i} + 1\,\mathbf{j}$ and $\mathbf{F}(t) = P(x,t)\,\mathbf{i} + Q(x,t)\,\mathbf{j}$. We have

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\int_{C_2'} \mathbf{F}\cdot d\mathbf{r} = \frac{\partial}{\partial y}\int_{y_1}^{y} Q(x,t)\,dt = Q(x,y).$$

Therefore, we know that

$$\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j} = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = \nabla f.$$

□

□ 由 **Theorem 7** 知: 第二類曲線積分與路徑選取無關 ⇒ 保守向量場。

Next, we will determine whether or not a vector field $\mathbf{F}$ is conservative. Suppose that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative, where P and Q have continuous first order partial derivatives. Then there is a function f such that $\mathbf{F} = \nabla f$, that is, $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$. By Clairaut's Theorem, we know

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

Hence $Q_x = P_y$ is a necessary condition (必要條件) that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative.

Theorem 8 (page 1090)**.** *If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first order partial derivatives on a domain D, then throughout D we have*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

The condition $Q_x = P_y$ is a sufficient condition *for a simply connected region.*

Definition 9 (page 1090)**.**

(a) We say C is a *simple curve* (簡單曲線) if it doesn't intersect itself anywhere between its endpoints. ($\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$).

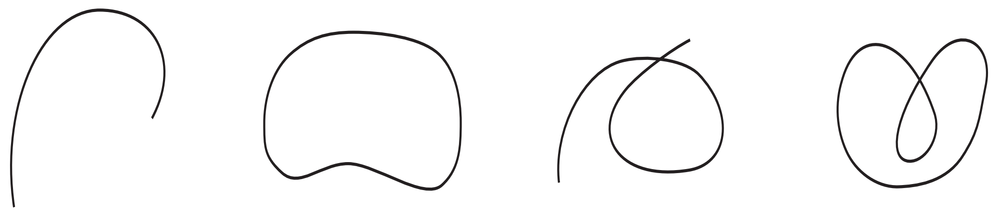

Figure 5: (Left to right) Simple, not closed; simple closed; not simple, not closed; not simple, closed.

(b) D is a *simply connected region* (單連通區域) in a plane if it is path connected and every simple closed curve in D enclosed only points that are in D.

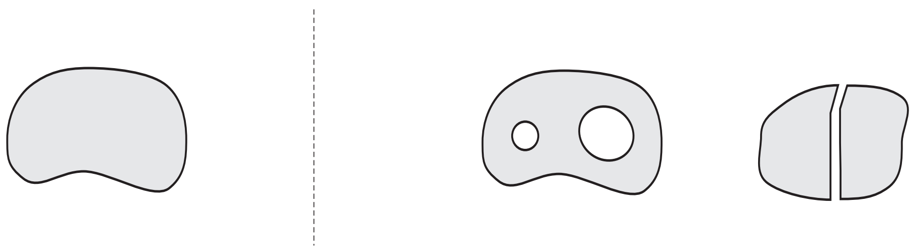

Figure 6: Simply connected region; non simply connected region.

Theorem 10 (page 1091)**.** *Let* $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ *be a vector field on an open simply-connected region* D*. Suppose that* P *and* Q *have continuous first-order derivatives and*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \quad \textit{throughout} \quad D,$$

then $\mathbf{F}$ *is conservative.*

We will prove Theorem 10 in the next section.

☐ 直觀上, 單連通區域代表此區域沒有「洞」– 虧格爲零 (genus); 而且無法分成兩塊。

Finally, we will use "partial integration" to find the potential functions.

Example 11 (page 1091)**.**

(a) If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = \mathrm{e}^t \sin t\,\mathbf{i} + \mathrm{e}^t \cos t\,\mathbf{j}$, and t from 0 to π.

Solution.

Example 12 (page 1095)**.** Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Proof. Since $\mathbf{F} = \nabla f$, we have $P = f_x, Q = f_y$, and $R = f_z$. By Clairaut's Theorem, we know that $P_y = (f_x)_y = f_{xy} = f_{yx} = (f_y)_x = Q_x$, $P_z = (f_x)_z = f_{xz} = f_{zx} = (f_z)_x = R_x$, and $Q_z = (f_y)_z = f_{yz} = f_{zy} = (f_z)_y = R_y$. ☐

Example 13. If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + \mathrm{e}^{3z})\,\mathbf{j} + 3y\mathrm{e}^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

Solution.

Example 14 (page 1095)**.** Let $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$.

(a) Show that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

(b) Show that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is *not* independent of path.

(c) Compute $\nabla\theta(x, y)$, where $\theta = \theta(x, y)$ is the polar angle function.

Solution.

Appendix: Conservation of Energy, page 1093

We will apply these ideas to a continuous force field $\mathbf{F}$ that moves an object along a path C given by $\mathbf{r}(t), a \leq t \leq b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point of C.

According to Newton's Second Law of Motion, the force $\mathbf{F}(\mathbf{r}(t))$ at a point on C is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation $\mathbf{F}(\mathbf{r}(t)) = m\,\mathbf{r}''(t)$, so the work done by the force on the object is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_a^b m\,\mathbf{r}''(t) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\ &= \frac{m}{2} \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}'(t) \cdot \mathbf{r}'(t))\,\mathrm{d}t = \frac{m}{2} \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}'(t)|^2\,\mathrm{d}t = \frac{m}{2} \Big[|\mathbf{r}'(t)|\Big]\Big|_a^b \\ &= \frac{m}{2}(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2) = \frac{1}{2}m|\mathbf{v}(b)|^2 - \frac{1}{2}m|\mathbf{v}(a)|^2, \end{aligned} \tag{3}$$

where $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity.

The quantity $\frac{1}{2}m|\mathbf{v}(t)|^2$, is called the *kinetic energy* (動能) of the object. Therefore we can rewrite Equation (3) as $W = K(B) - K(A)$, which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that $\mathbf{F}$ is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$. In physics, the *potential energy* (位能) of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = \nabla f = -\nabla P$. Then we have

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_C \nabla P \cdot \mathrm{d}\mathbf{r} = -(P(\mathbf{r}(b)) - P(\mathbf{r}(a))) = P(A) - P(B).$$

Comparing this equation with $W = K(B) - K(A)$, we see that

$$P(A) + K(A) = P(B) + K(B),$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the *Law of Conservation of Energy* (能量守恆定律) and it is the reason the vector field is called *conservative*.