16.2 Line Integrals, page 1075

Line Integrals of Scalar Functions (第一類曲線積分), page 1075

Suppose that a smooth plane curve C is given by the parametric equations $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, a \le t \le b$. Recall that a curve is smooth means that $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. We will define an integral over a curve C.

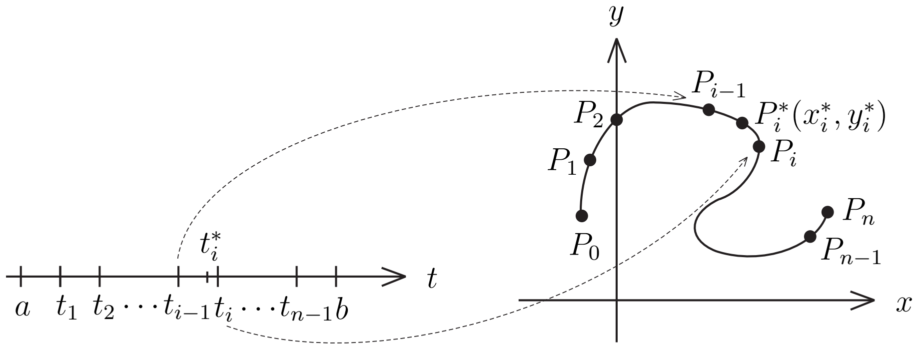

Figure 1: Line integrals of scalar functions.

(a) We divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width. We let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$.

(b) Choose any point $P_i(x_i^*, y_i^*)$ (corresponding to $t_i^* \in [t_{i-1}, t_i]$) in the i-th subarc.

(c) If $f(x, y)$ is any function of two variables whose domain includes the curve C, we form the sum (similar to a Riemann sum) $\sum\limits_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$.

(d) Define the *line integral of $f(x, y)$ along C* (函數 $f(x, y)$ 沿曲線 C 的線積分) is

$$\int_C f(x, y)\,\mathrm{d}s = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

Recall that the arc length function of C is $s(t) = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2}\,\mathrm{d}u$, and it implies $\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t$, so the line integral has the following expression:

$$\int_C f(x, y)\,\mathrm{d}s = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t.$$

Well-defined problem: (copy from Wikipedia) In mathematics, an expression is called *well-defined* (良好定義的) if its definition assigns it a unique interpretation or value. For example, *a function is well-defined* if it gives the same result when the representation of the input is changed without changing the value of the input. That is, if $f(x)$ is a well-defined function defined on $\mathbb{R}$, then $f(0.5)$ must be equal to $f(\frac{1}{2})$, and $f(1)$ must be equal to $f(0.\bar{9})$.

Suppose that $\mathbf{r}(v) = x(v)\,\mathbf{i} + y(v)\,\mathbf{j}, c \le v \le d$ is another parametrization of the plane curve C. To show the line integral is well-defined, we have to check that

$$\int_C f(x,y)\,\mathrm{d}s = \int_c^d f(x(v),y(v))\sqrt{(x'(v))^2 + (y'(v))^2}\,\mathrm{d}v.$$

<u>Check</u>: This is because the arc length is $s(v) = \int_a^v \sqrt{(x'(u))^2 + (y'(u))^2}\,\mathrm{d}u$, and it implies $\mathrm{d}s = \sqrt{(x'(v))^2 + (y'(v))^2}\,\mathrm{d}v$.

The geometric meaning of the line integrals is to compute the area of one side of the "fence" or "curtain," whose base is C and whose "height" at (x,y) is $f(x,y)$.

☐ $\mathrm{d}s$ 是弧長參數, 爲正量, 給定曲線參數式後, 上、下限代入終點與起點。

☐ 第一類曲線積分與「路徑的方向無關」, 所以化爲定積分時, 下限總是小於上限。

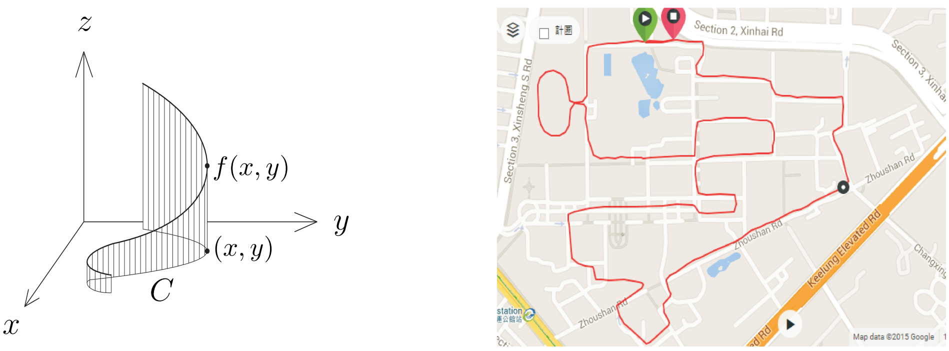

Figure 2: Line integrals: The area of the fence. The total calories of running.

Example 1 (page 1076)**.** Evaluate $\int_C (2 + x^2y)\,\mathrm{d}s$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

Solution.

We can define the line integrals on *piecewise smooth curve* (分段光滑曲線) C, which is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, and the initial point of C_{i+1} is the terminal point of C_i. The line integral of $f(x, y)$ along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)\,\mathrm{d}s = \int_{C_1} f(x,y)\,\mathrm{d}s + \int_{C_2} f(x,y)\,\mathrm{d}s + \cdots + \int_{C_n} f(x,y)\,\mathrm{d}s.$$

Example 2 (page 1077)**.** Evaluate $\int_C 2x\,\mathrm{d}s$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

Solution.

Any physical interpretation of $\int_C f(x, y)\,\mathrm{d}s$ depends on the physical interpretation of the function f. For example, we define the *mass* (質量) m of the wire:

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\Delta s_i = \int_C \rho(x,y)\,\mathrm{d}s.$$

The *center of mass* (質心) of the wire with density function $\rho(x, y)$ is

$$(\bar{x}, \bar{y}) = \left(\frac{1}{m}\int_C x\rho(x,y)\,\mathrm{d}s, \frac{1}{m}\int_C y\rho(x,y)\,\mathrm{d}s\right).$$

Example 3 (page 1077)**.** A wire takes the shape of the semicircle $x^2+y^2 = 1, y \geq 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Solution.

Definition 4 (page 1066)**.**

(a) Define *line integrals of f along C with respect to x* (f 沿 C 對於 x 的線積分):

$$\int_C f(x,y)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i = \int_a^b f(x(t), y(t))x'(t)\,\mathrm{d}t.$$

(b) Define *line integrals of f along C with respect to y* (f 沿 C 對於 y 的線積分):

$$\int_C f(x,y)\,\mathrm{d}y = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i = \int_a^b f(x(t), y(t))y'(t)\,\mathrm{d}t.$$

(c) The *line integral of f along C with respect to arc length* is:

$$\int_C f(x,y)\,\mathrm{d}s = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t.$$

☐ 函數 f 沿 C 對於 x 的線積分, 因爲 $\mathrm{d}x$ 可能正可能負, 因此積分與路徑的方向有關。

It frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to abbreviate by writing

$$\int_C P(x,y)\,\mathrm{d}x + \int_C Q(x,y)\,\mathrm{d}y = \int_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y.$$

Example 5. Evaluate $\int_C y^2\,\mathrm{d}x + x\,\mathrm{d}y$, where

(a) $C = C_1$ is the line segment from $(-3,-2)$ to $(0,1)$.

(b) $C = C_2$ is the arc of the parabola $x = 1 - y^2$ from $(-3,-2)$ to $(0,1)$.

Solution.

Line Integrals in Space, page 1080

Suppose that C is a smooth space curve given by the parametric equation $x = x(t), y = y(t), z = z(t), a \le t \le b$ or vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. If $f(x, y, z)$ is a smooth function that is continuous on some region containing C, then we define the *line integral of f along C* as

$$\begin{aligned}\int_C f(x,y,z)\,\mathrm{d}s &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i \\ &= \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))}\,\mathrm{d}t \\ &= \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,\mathrm{d}t.\end{aligned}$$

Line integrals along C with respect to $x, y,$ and z can also be defined. For example,

$$\int_C f(x,y,z)\,\mathrm{d}z = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta z_i = \int_a^b f(x(t),y(t),z(t))z'(t)\,\mathrm{d}t.$$

Therefore, we evaluate integrals of the form

$$\int_C P(x,y,z)\,\mathrm{d}x + Q(x,y,z)\,\mathrm{d}y + R(x,y,z)\,\mathrm{d}z$$

by expressing everything $(x, y, z, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ in terms of the parameter t.

Line Integrals of Vector Fields (第二類曲線積分), page 1082

Suppose that $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$ such as gravitational field, electric force field, etc. We want to *compute the work* (功) *done by this force* in moving a particle along a smooth curve C.

(a) We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width.

(b) Choose $P_i^*(x_i^*, y_i^*, z_i^*)$ on the i-th subarc corresponding to t_i^*.

(c) If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction $\mathbf{T}(t_i^*)$, the *unit* tangent vector at P_i^*. Thus the total work done by the force $\mathbf{F}$ in moving the particle along C is approximately $\sum_{i=1}^{n} \mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)\Delta s_i$.

(d) When n tends to infinity, we define the *work* (功) W done by the force field $\mathbf{F}$:

$$W = \int_C \mathbf{F}(x,y,z)\cdot\mathbf{T}(x,y,z)\,\mathrm{d}s = \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s. \qquad (1)$$

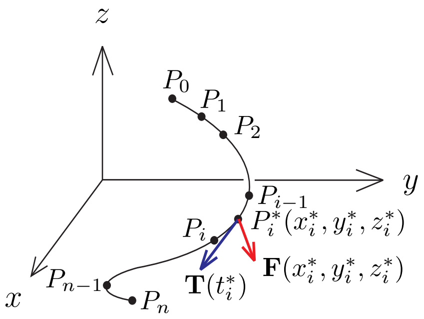

Figure 3: Line integrals of vector fields.

Hence, *work is the line integral with respect to arc length of the tangential component of the force.*

If the curve C is given by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, then we get the unit tangent vector is $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, so we can rewrite (1) in the form

$$\begin{aligned} W &= \int_C \left(\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right) |\mathbf{r}'(t)| \, \mathrm{d}t = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, \mathrm{d}t \\ &= \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F}(x(t), y(t), z(t)) \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}. \end{aligned}$$

Definition 6 (page 1082)**.** Let $\mathbf{F}$ be a continuous vector field on a smooth curve C given by a vector function $\mathbf{r}(t), a \le t \le b$. Then the *line integral of* $\mathbf{F}$ *along* C is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, \mathrm{d}t.$$

- ☐ 第二類曲線積分與積分路徑的方向有關 (方向相反時, 積分值變號)。
- ☐ 積分路徑的方向會用 C 與 $-C$ 表示, 而 $\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$。

Connection between line integrals of vector fields and line integral of scalar functions, page 1084

Suppose that the vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given in component form by the equation $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$. Then

$$\begin{aligned} &\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, \mathrm{d}t = \int_a^b (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot (x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}) \, \mathrm{d}t \\ &= \int_a^b (P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)) \, \mathrm{d}t \\ &= \int_C P(x, y, z) \, \mathrm{d}x + Q(x, y, z) \, \mathrm{d}y + R(x, y, z) \, \mathrm{d}z. \end{aligned}$$

Example 7. Find the work done by the force field $\mathbf{F}(x, y) = -y^2\,\mathbf{i} + x^2\,\mathbf{j}$ in moving a particle along arc of the circle $x^2 + y^2 = 2$ traversed counterclockwise from $(\sqrt{2}, 0)$ to $(-\sqrt{2}, 0)$.

Solution.

Example (TA) 8. Evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where $\mathbf{F} = -3z\,\mathbf{i} + \frac{3}{2}x\,\mathbf{j}$ and C is the space curve which is the intersection of $x^2 + y^2 + z^2 = 1$ and $y = 1 - x^2$ in the first octant from $(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2})$ to $(1, 0, 0)$.

Solution.