Chapter 16 Vector Calculus

16.1 Vector Fields, page 1068

Definition 1 (page 1069)**.**

(1) Let D be a set in $\mathbb{R}^2$. A *vector field on* $\mathbb{R}^2$ (向量場) is a map $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

(2) Let E be a subset of $\mathbb{R}^3$. A *vector field on* $\mathbb{R}^3$ is a map $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y) for a few representative points in D.

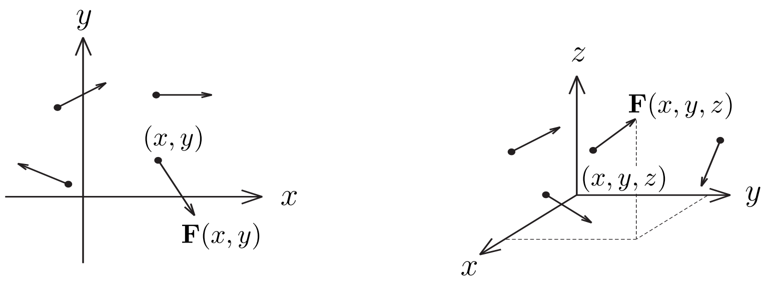

Figure 1: Vector fields on $\mathbb{R}^2$ and on $\mathbb{R}^3$.

Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its *component functions* (分量函數) P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = (P(x, y), Q(x, y)).$$

Functions $P(x, y)$ and $Q(x, y)$ are called *scalar function* (純量函數) or *scalar fields.*

Example 2 (page 1070)**.** A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Denote $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j}$ by the position vector.

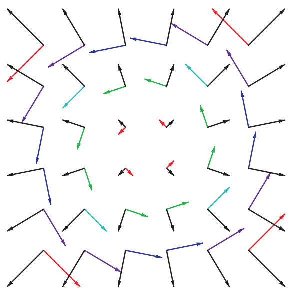

Figure 2: Vector Fields $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} = (-y, x)$ and $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} = (x, y)$.

Remark that $\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) = (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j}) = -xy + xy = 0$, so two vector fields are orthogonal (正交).

Example 3 (page 1071)**.** Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{GMm}{r^2},$$

where r is the distance between the objects and G is the gravitational constant.

Let the position vector of the object with mass m be $\mathbf{x} = (x, y, z)$, then $r^2 = |\mathbf{x}|^2$. Therefore the gravitational force acting on the object at $\mathbf{x}$ is

$$\mathbf{F}(\mathbf{x}) = -\frac{GMm}{|\mathbf{x}|^2}\frac{\mathbf{x}}{|\mathbf{x}|} = -\frac{GMm}{|\mathbf{x}|^3}\mathbf{x}, \tag{1}$$

and we say the equation (1) is *gravitational field* (重力場).

Example 4 (page 1072)**.** Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ (or electric field 電場) exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = (x, y, z)$ is

$$\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Qq}{|\mathbf{x}|^2}\frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\varepsilon Qq}{|\mathbf{x}|^3}\mathbf{x},$$

where ε is a constant. For like charges, we have $Qq > 0$ and the force is repulsive; for unlike charges, we have $Qq < 0$ and the force is attractive.

Instead of considering the electric force $\mathbf{F}$, physicists often consider the force per unit charge (電場強度):

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}.$$

Gradient Fields (梯度場), page 1072

Recall that for a smooth function $f(x, y)$, the gradient ∇f, or $\operatorname{grad} f$, is defined by

$$\nabla f(x, y) = \operatorname{grad} f = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}.$$

Likewise, if $f(x, y, z)$ is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by $\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$.

Definition 5 (page 1072)**.**

(a) For a scalar function f, we say ∇f is a *gradient vector field* (梯度向量場).

(b) A vector field $\mathbf{F}$ is called a *conservative vector field* (保守向量場) if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a *potential function* (位勢函數) for $\mathbf{F}$.

Not all vector fields are conservative, but such fields do arise frequently in physics.

Example 6 (page 1073)**.** The gravitational field $\mathbf{F}$ is conservative because if we define a function

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}},$$

then

$$\nabla f =$$

Example 7. Let $f(x, y)$ be a smooth function, then the gradient vector field $\nabla f(x, y)$ is perpendicular to the level curves $f(x, y) = k$.

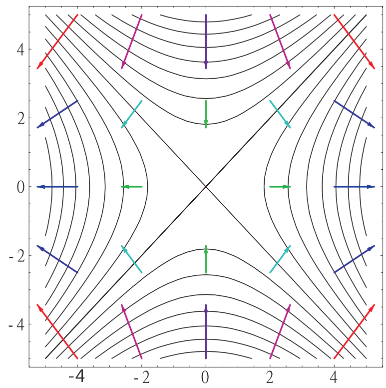

Figure 3: Level sets of $f(x, y) = x^2 - y^2$ and the gradient field $\nabla f = 2x\,\mathbf{i} - 2y\,\mathbf{j}$.

In general, all conservative vector vector field $\mathbf{F}$ is perpendicular to the level sets of its potential function f.

Exercise (page 1074)**.** Match the functions f_1, f_2, f_3, and f_4 with the plots of their gradient vector fields labeled `I - IV`. Give reasons for your choices.

(a) $f_1(x, y) = x^2 + y^2$.

(b) $f_2(x, y) = x(x + y)$.

(c) $f_3(x, y) = (x + y)^2$.

(d) $f_4(x, y) = \sin \sqrt{x^2 + y^2}$.

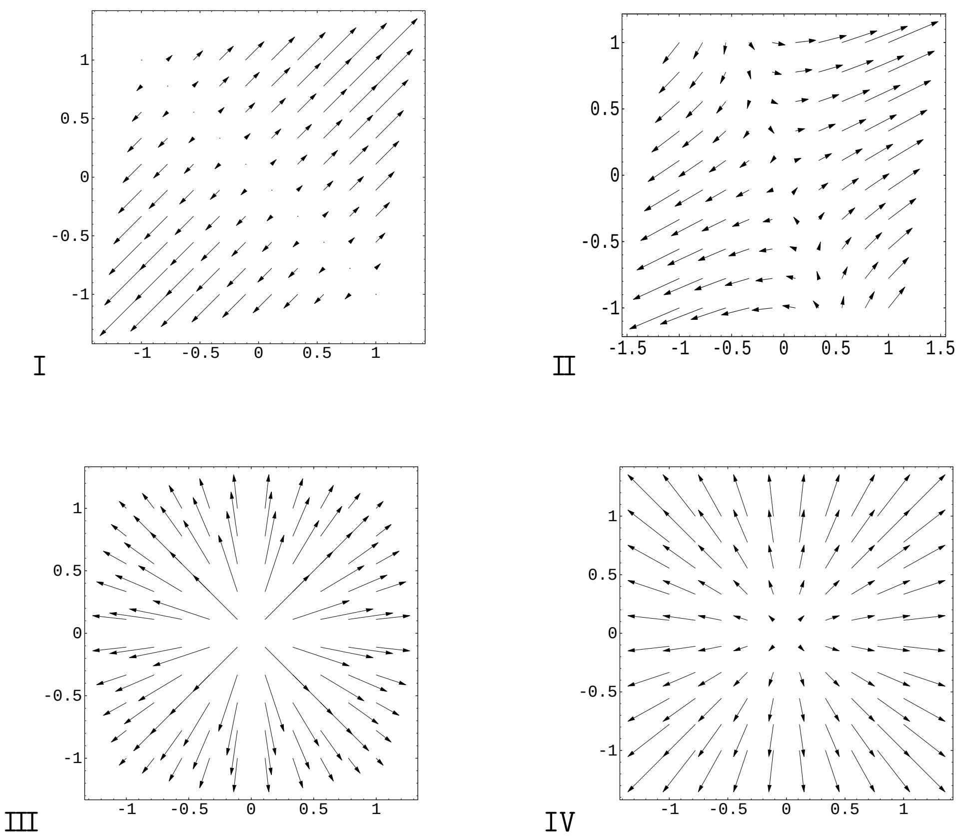

Figure 4: Gradient vector fields.