15.9 Change Variables in Multiple Integrals, page 1052

<u>**Goal**</u>: Find relations of change of variable in double and triple integrals.
Recall that

(1) For a function of one variable $f(x)$, we have the Substitution Rule:

$$\int_a^b f(x)\,\mathrm{d}x = \int_c^d f(x(u))x'(u)\,\mathrm{d}u,$$

where $x = x(u)$ and $a = x(c), b = x(d)$.

(2) In section 15.4, we get the formula of double integrals in polar coordinates. Supppose that $x = r\cos\theta$, $y = r\sin\theta$, then

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_S f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r\,\mathrm{d}\theta,$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

More generally, we consider a change of variables that is given by a C^1 and one-to-one *transformation* T form the uv-plane to the xy-plane (一次偏導數連續且一對一的坐標變換):

$$T(u,v) = (x,y),$$

where x and y are related to u and v by the equations

$$T : \begin{cases} x = x(u,v) \\ y = y(u,v), \end{cases} \qquad T^{-1} : \begin{cases} u = u(x,y) \\ v = v(x,y). \end{cases}$$

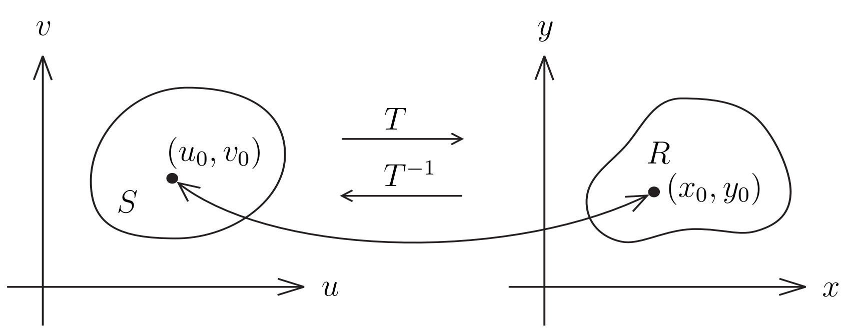

Figure 1: Transformation T and inverse transformation T^{-1}.

Definition 1 (page 1053)**.**

(a) The terminology C^1 means that $x(u, v)$ and $y(u, v)$ have continuous first-order partial derivatives.

(b) If $T(u_1, v_1) = (x_1, y_1)$, then (x_1, y_1) is called the *image* of (u_1, v_1).

(c) T is called *one-to-one* if no two points have the same image.

(d) T transforms S into a region R in the xy-plane called the *image of* S, consisting of the images of all points in S.

Example 2 (page 1053)**.** A transformation is defined by the equations $x = u^2 - v^2, y = 2uv$. Find the image of the square $S = \{(u, v)|0 \leq u \leq 1, 0 \leq v \leq 1\}$.

Solution.

Now we will see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower corner is the point (u_0, v_0) and whose dimensions are Δu and Δv. The image of S is a region R in the xy-

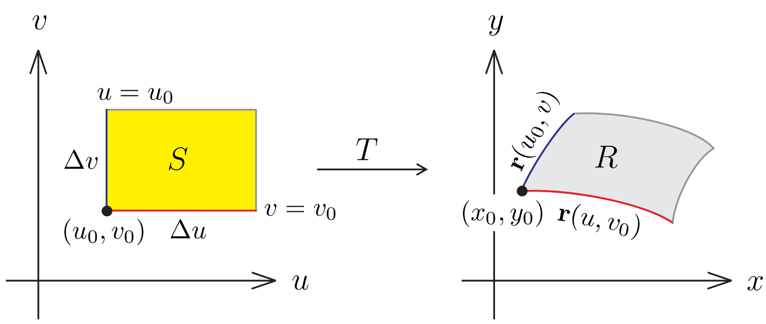

Figure 2: Transformation T from a rectangle S to a region R.

plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector $\mathbf{r}(u, v) = x(u, v)\,\mathbf{i}+y(u, v)\,\mathbf{j}$ is the position vector of the image of the point (u, v). The equation of the lower side of S is $v = v_0$, whose image curve is given by the vector function $\mathbf{r}(u, v_0)$. The tangent vector at (x_0, y_0) to this image curve is

$$\mathbf{r}_u = x_u(u_0, v_0)\,\mathbf{i} + y_u(u_0, v_0)\,\mathbf{j}$$

Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely, $u = u_0$) is

$$\mathbf{r}_v = x_v(u_0, v_0)\,\mathbf{i} + y_v(u_0, v_0)\,\mathbf{j}$$

We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \text{and} \qquad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0).$$

Since

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

and so

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \mathbf{r}_u \quad \text{and} \quad \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \mathbf{r}_v.$$

This means that we can approximate R by a parallelogram determined by the vectors $\Delta u \mathbf{r}_u$ and $\Delta v \mathbf{r}_v$. Therefore we can approximate the area of R by the area of this parallelogram

$$|(\Delta u)\mathbf{r}_u \times (\Delta v)\mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v|\Delta u \Delta v = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v.$$

Definition 3 (page 1055)**.** The *Jacobian* of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

With this notation we can get $\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$, where the Jacobian is evaluated at (u_0, v_0).

For the general region S in the uv-plane we divide S into rectangles S_{ij} and call their images in the xy-plane R_{ij}. Applying the approximation to each R_{ij}, we approximate the double integral of f over R as follows:

$$\begin{aligned}\iint_R f(x, y)\,\mathrm{d}A &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_i)\Delta A \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x(u_i, v_i), y(u_i, v_i)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right|_{(u_i, v_i)} \Delta u \Delta v + \text{H.O.T.} \\ &= \iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

Change to Variables in a Double Integral (page 1056)**.** *Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then*

$$\iint_R f(x, y)\,\mathrm{d}A = \iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v.$$

Example 4 (page 1058)**.** Evaluate the integral $\iint_R \mathrm{e}^{\frac{x+y}{x-y}}\,\mathrm{d}A$, where R is the trapezoidal region with vertices $(1, 0), (2, 0), (0, -2)$, and $(0, -1)$.

Solution.

Example 5 (page 1057)**.** Use $x = u^2 - v^2, y = 2uv$ to evaluate the integral $\iint_R y\,\mathrm{d}A$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x, y \geq 0$.

Solution.

Example 6. Evaluate $\iint_{x^2+xy+y^2\le 1} \mathrm{e}^{-(x^2+xy+y^2)}\,\mathrm{d}A$.

Solution.

Triple Integrals, page 1059

The *Jacobian* of the transformation T is the following 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

We have the following formula for triple integrals:

$$\iiint_R f(x,y,z)\,\mathrm{d}V = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

Example 7. Elliptic cylindrical coordinate system is

$$x = ar\cos\theta, \qquad y = br\sin\theta, \qquad \tilde{z} = cz,$$

where $a,b,c>0$ are constants. The volume element is ____________________.

Example 8. Ellipsoidal coordinate system is

$$x = a\rho\sin\phi\cos\theta, \qquad y = b\rho\sin\phi\sin\theta, \qquad z = c\rho\cos\phi,$$

where $a,b,c>0$ are constants. The volume element is ____________________.

Appendix

Suppose that $f(x)\in C^1([a,b])$, which implies $|f'(x)|\le M$. Let $\Delta x = \frac{b-a}{n}$, then

$$\left|\int_a^b f(x)\,\mathrm{d}x - \sum_{i=1}^n f(x_i^*)\Delta x\right| \le \sum_{i=1}^n \left|\max_{[x_{i-1},x_i]} f(x) - \min_{[x_{i-1},x_i]} f(x)\right| \Delta x$$
$$\le \sum_{i=1}^n |f'(\xi_i)|(\Delta x)^2 = M\sum_{i=1}^n \frac{(b-a)^2}{n^2} = M\cdot\frac{(b-a)^2}{n} \to 0 \text{ as } n\to\infty.$$

So for integration, before we take summation, the $\frac{1}{n}$ part is the whole material. We can ignore higher order term such as $\frac{1}{n^2}$ because it tends to zero after summation and n tends to infinity.

Exercise. Evaluate the integral $\iint_R \sin(x+y)\cos(2x-y)\,\mathrm{d}A$, where R is the region bounded by $y = 2x-1, y = 2x+3, y = -x$, and $y = -x+1$.

Exercise. Evaluate the integral $\iiint_E (x^2y + 3xyz)\,\mathrm{d}V$, where E is the region $1 \le x \le 2, 0 \le xy \le 2$, and $0 \le z \le 1$.

Exercise. Evaluate the integral $\iint_R \mathrm{e}^{x^2-xy+y^2}\,\mathrm{d}A$, where $R = \{(x,y)|x^2 - xy + y^2 \le a^2\}$.

Exercise. Evaluate the integral $\iint_R \sin(x^2+2xy+y^2)\,\mathrm{d}A$, where R is the region bounded by $x + y = 0, x = 0$, and $y = 0$.

Exercise. Evaluate the integral $\iint_R \frac{x^2}{x^2+4y^2}\,\mathrm{d}A$, where R is the region bounded by two ellipses $x^2 + 4y^2 = 1$ and $x^2 + 4y^2 = 4$.

Exercise. Evaluate the integral $\int_0^{\frac{3}{2}} \int_y^{1-\frac{y}{2}} (2x+y)\mathrm{e}^{y-x}\,\mathrm{d}x\,\mathrm{d}y$.

Exercise. Evaluate the integral $\iiint_E (x+y+z)^2\,\mathrm{d}V$, where $E = \{(x,y,z)|2x^2 + 3y^2 + 5z^2 + 6yz + 2xz \le 1\}$.

Exercise. Compute the area of the domain in the first quadrant bounded by the four curves $xy = 1, xy = 4, \frac{y}{x^2} = 1$, and $\frac{y}{x^2} = 2$.

Exercise. Find the region $E \subset \mathbb{R}^3$ for which the triple integral $\iiint_E (4 - x^2 - 4y^2 - 9z^2)\,\mathrm{d}V$ is a maximum, and compute this maximum value.

Exercise. Evaluate $\iint_{x^2+xy+y^2 \le 1} \mathrm{e}^{-(x^2+xy+y^2)}\,\mathrm{d}A$

Exercise. Find $\iiint_E xyz\,\mathrm{d}V$, where $E = \{(x,y,z)|x \ge 0, y \ge 0, \ge 0, 36x^2 + 16y^2 + 9z^2 \le 144\}$.

Exercise. Evaluate $\iint_R \sin\left(\frac{2y-x}{2y+x}\right)\,\mathrm{d}A$, where R is the region bounded by $2y + x = 1, 2y + x = 2, 2y - x = 0$, and $2y + 5x = 0$.

Exercise. Evaluate $\iint_R \mathrm{e}^{-4x^2+12xy-10y^2}\,\mathrm{d}A$, where R is the region satisfying $x \ge 2y$ and $y \ge 0$.

Exercise. Evaluate $\iint_R \mathrm{e}^{-4x^2-9y^2}\,\mathrm{d}A$, where R is the region satisfying $2x \le 3y$ and $x \ge 0$.

Exercise. Evaluate $\iint_R \mathrm{e}^{xy}\,\mathrm{d}A$, where R is the region bounded by $xy = 1, xy = 4, y = 1$, and $y = 3$.

Exercise. Evaluate the double integral $\displaystyle\iint_R (x+y)^2\sin^2(x-y)\,\mathrm{d}A$, where R is the square region with vertices $(\frac{\pi}{2}, 0), (\pi, \frac{\pi}{2}), (\frac{\pi}{2}, \pi)$, and $(0, \frac{\pi}{2})$.