15.8 Triple Integrals in Spherical Coordinates, page 1045

Goal: Define and compute triple integrals in spherical coordinates.

Spherical Coordinates, page 1045

The *spherical coordinates* (ρ, θ, ϕ) (球坐標) of a point P in space are shown in Figure 1, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that we assume $\rho \geq 0$ and $0 \leq \phi \leq \pi$.

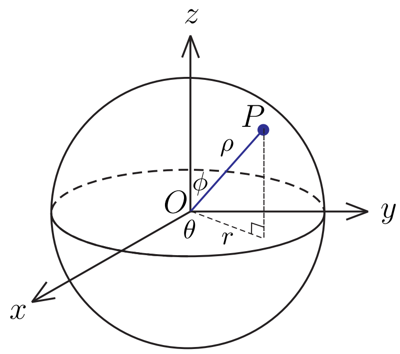

Figure 1: Spherical coordinate system.

☐ 不同的書籍或文章會用不同的記號 (有的用 r 而非 ρ) 與定義方式 (ϕ 的取法不同)。

The spherical coordinate system is useful in problems where there is symmetry about a point, and the origin is placed at this point. Figure 2 shows the surfaces of $\rho = c, \theta = c$, and $\phi = c$.

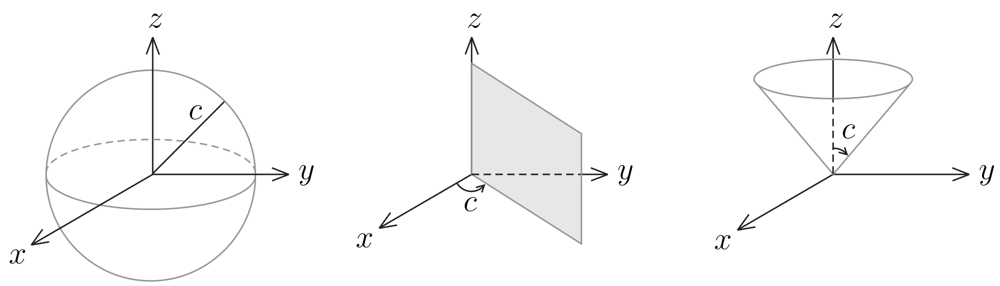

Figure 2: (a) $\rho = c$ is a sphere. (b) $\theta = c$ is a half-plane. (c) $\phi = c$ is a half-cone.

Relations between rectangular coordinates and spherical coordinates are

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi.$$
$$\rho^2 = x^2 + y^2 + z^2, \qquad \tan\theta = \frac{y}{x}, \qquad \tan^2\phi = \frac{x^2+y^2}{z^2}.$$

Example 1 (page 1046)**.** The point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Solution.

Example 2 (page 1046)**.** The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

Solution.

Evaluating Triple Integrals with Spherical Coordinates, page 1047

Consider the *spherical wedge* $E = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$, where $a \geq 0$ and $\beta - \alpha \leq 2\pi$, and $d - c \leq \pi$.

(1) Divide E equally into E_{ijk} by $\rho = \rho_i, \theta = \theta_j$, and $\phi = \phi_k$.

(2) E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i \Delta\phi$, and $\rho_i \sin \phi_k \Delta\theta$. So an approximation to the volume of E_{ijk} is given by

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i \Delta\phi)(\rho_i \sin \phi_k \Delta\theta) = \rho_i^2 \sin \phi_k \Delta\rho\Delta\theta\Delta\phi.$$

In fact, by the Mean Value Theorem (see the Appendix), the volume of E_{ijk} is given exactly by

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin \tilde{\phi}_k \Delta\rho\Delta\theta\Delta\phi,$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k)$ is some point in E_{ijk}. Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be the rectangular coordinates of the sample point $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k)$.

(3) We get the Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\tilde{\rho}_i \sin \tilde{\phi}_k \cos \tilde{\theta}_j, \tilde{\rho}_i \sin \tilde{\phi}_k \sin \tilde{\theta}_j, \tilde{\rho}_i \cos \phi_j) \tilde{\rho}_i^2 \sin \tilde{\phi}_k \Delta\rho\Delta\theta\Delta\phi.$$

(4) When $l, m, n \to \infty$, we get the *formula for triple integration in spherical coordinates*:

$$\iiint_E f \, \mathrm{d}V = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

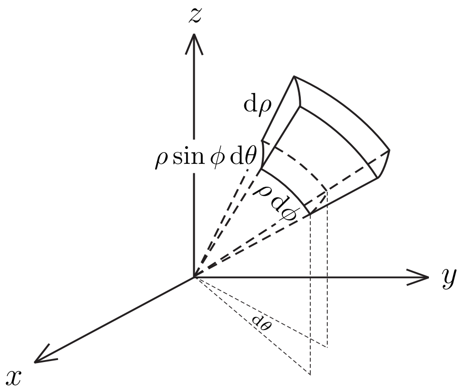

Figure 3: Volume element of the spherical coordinate system.

This formula can be extended to include more general spherical regions such as $E = \{(\rho, \theta, \phi) | \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\}$, and in this case the triple integration will be

$$\iiint_E f \, \mathrm{d}V = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

Example 3 (page 1048)**.** Evaluate $\displaystyle\iiint_B \mathrm{e}^{(x^2+y^2+z^2)^{\frac{3}{2}}} \, \mathrm{d}V$, where B is the unit ball: $B = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$.

Solution.

Example 4 (page 1048)**.** Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Solution.

Appendix (Proof of the following statement)

"For all spherical wedge E, there exists $(\tilde{\rho}, \tilde{\theta}, \tilde{\phi})$ in E so that $\Delta V = \tilde{\rho}^2 \sin\tilde{\phi}\Delta\rho\Delta\theta\Delta\phi$."

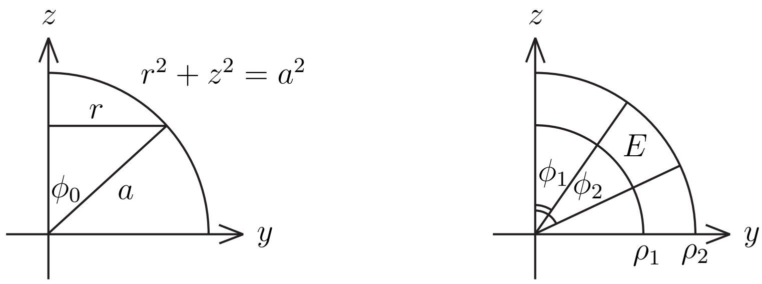

Figure 4: Volume element of the spherical coordinate system.

(a) Show that "the volume of the solid bounded above by the sphere $r^2 + z^2 = a^2$ and below by the cone $z = r\cot\phi_0, 0 < \phi_0 < \frac{\pi}{2}$, and $\theta_1 \le \theta \le \theta_2$, is $V = \frac{a^3\Delta\theta}{3}(1 - \cos\phi_0)$, where $\Delta\theta = \theta_2 - \theta_1$". Using the cylindrical coordinates,

$$\begin{aligned}
V &= \int_{\theta_1}^{\theta_2}\int_0^{a\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{a^2-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \Delta\theta\int_0^{a\sin\phi_0}\left(r\sqrt{a^2-r^2} - r^2\cot\phi_0\right)\mathrm{d}r \\
&= \frac{\Delta\theta}{3}\left[-(a^2-r^2)^{\frac{3}{2}} - r^3\cot\phi_0\right]\Big|_{r=0}^{r=a\sin\phi_0} \\
&= \frac{\Delta\theta}{3}\left(-(a^2 - a^2\sin^2\phi_0)^{\frac{3}{2}} + a^3 - a^3\sin^3\phi_0\cot\phi_0\right) \\
&= \frac{a^3\Delta\theta}{3}\left(1 - \cos^3\phi_0 - \sin^2\phi_0\cos\phi_0\right) = \frac{a^3\Delta\theta}{3}(1 - \cos\phi_0).
\end{aligned}$$

(b) Show that the volume of the spherical wedge given by $\rho_1 \le \rho \le \rho_2, \theta_1 \le \theta \le \theta_2, \phi_1 \le \phi \le \phi_2$ is $\Delta V = \frac{\Delta\theta}{3}(\rho_2^3 - \rho_1^3)(\cos\phi_1 - \cos\phi_2)$. Denote V_{ij} by the volume of the region bounded by the sphere of radius ρ_i and the cone with angle ϕ_j, and θ from θ_1 to θ_2. Then we have

$$\begin{aligned}
V &= (V_{22} - V_{21}) - (V_{12} - V_{11}) \\
&= \frac{\Delta\theta}{3}(\rho_2^3(1-\cos\phi_2) - \rho_2^3(1-\cos\phi_1) - \rho_1^3(1-\cos\phi_2) + \rho_1^3(1-\cos\phi_1)) \\
&= \frac{\Delta\theta}{3}(\rho_2^3 - \rho_1^3)(\cos\phi_1 - \cos\phi_2).
\end{aligned}$$

(c) By the Mean Value Theorem with $f(\rho) = \rho^3$, there exists some $\tilde{\rho} \in (\rho_1, \rho_2)$ such that $f(\rho_2) - f(\rho_1) = f'(\tilde{\rho})(\rho_2 - \rho_1) \Rightarrow \rho_2^3 - \rho_1^3 = 3\tilde{\rho}^2\Delta\rho$. Similarly, for $g(\phi) = \cos\phi$, there exists $\tilde{\phi} \in (\phi_1, \phi_2)$ such that $g(\phi_2) - g(\phi_1) = g'(\tilde{\phi})(\phi_2 - \phi_1) \Rightarrow \cos\phi_1 - \cos\phi_2 = \sin\tilde{\phi}\Delta\phi$. Hence for each spherical wedge E, there exists $(\tilde{\rho}, \tilde{\theta}, \tilde{\phi})$ in E such that

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin\tilde{\phi}_k\Delta\rho\Delta\theta\Delta\phi.$$