15.7 Triple Integrals in Cylindrical Coordinates, page 1040

<u>**Goal**</u>: Compute triple integrals in cylindrical coordinates.

Cylindrical Coordinates, page 1040

In the *cylindrical coordinate system* (柱坐標系), a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the distance from P to the xy-plane.

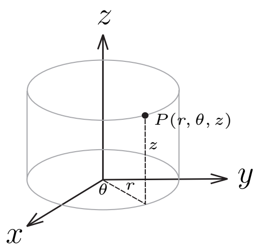

Figure 1: Cylindrical coordinate system.

- Relation from cylindrical to rectangular: $x = r\cos\theta, y = r\sin\theta, z = z$.
- Relation from rectangular to cylindrical: $r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}, z = z$.

Example 1 (page 1041)**.**

(a) Find rectangular coordinates of the point with cylindrical coordinates $(2, \frac{2}{3}\pi, 1)$.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Solution.

(a) Since $x =$ ______________________, $y =$ __________________, $z =$ ___. the point is _____________ in rectangular coordinates.

(b) Since $r =$ _________________, $\tan\theta =$ ____________________, $z =$ ___, one of cylindrical coordinates is ______________, and another is ________________. As with polar coordinates, there are infinitely many choices.

Example 2 (page 1041)**.** Describe the surface whose equation in cylindrical coordinates is $z = r$.

Solution. Since $z^2 = r^2 = x^2 + y^2$, it is a _____________ whose axis is the z-axis.

Evaluating Triple Integrals with Cylindrical Coordinates, page 1042

Suppose that E is a *type z* region whose projection D onto the xy-plane is conveniently described in polar coordinates. Suppose that $f(x, y, z)$ is continuous and

$$E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\},$$

where D is given in polar coordinates by $D = \{(r, \theta)|\alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$. We get *the formula for triple integration in cylindrical coordinates*:

$$\iiint_E f(x, y, z)\,\mathrm{d}V = \int_\alpha^\beta \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=u_1(r\cos\theta, r\sin\theta)}^{z=u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

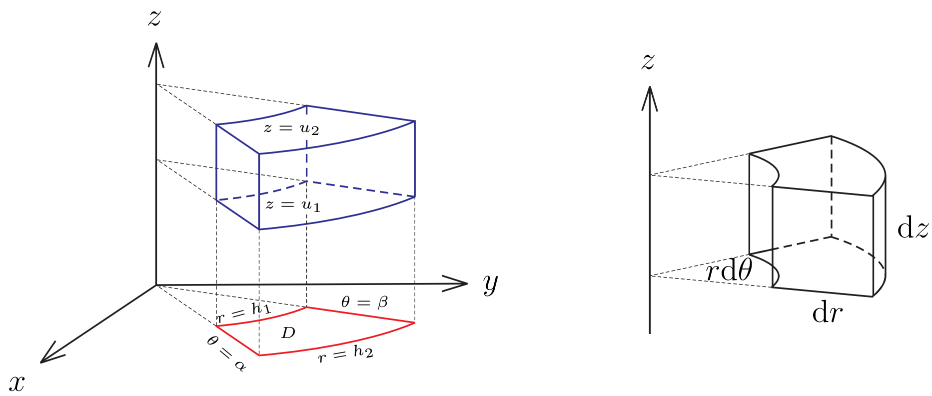

Figure 2: Cylindrical coordinate system.

Example 3. Evaluate $A = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$

Solution.

Homework (page 1044)**.** Find the volume of the solid enclosed by the three cylinders $x^2 + y^2 = 1, x^2 + z^2 = 1$, and $y^2 + z^2 = 1$.

Solution. The volume is ____________________.

We leave the calculation of this integral as an Exercise.