15.6 Triple Integrals, page 1029

<u>Goal</u>: Define and compute triple integrals of $f(x, y, z)$ over a bounded region.

We first deal with the case where $f(x, y, z)$ is defined on a rectangular box:

$$B = \{(x, y, z) | a \le x \le b, c \le y \le d, r \le z \le s\}.$$

(1) Divide the box B into lmn sub-boxes: $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$. Each B_{ijk} has area $\Delta V = \Delta x \Delta y \Delta z$.

(2) Choose a *sample point* $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each B_{ijk}.

(3) We get the *triple Riemann sum* (三重黎曼和): $\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$.

(4) **Definition 1** (page 1030). The *triple integral* (三重積分) of f over B is

$$\iiint_B f(x, y, z)\, \mathrm{d}V = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$$

if this limit exists.

☐ 三變數函數在三度空間中無法再畫成函數的圖形 (必須放到四度空間, 但不好想像)。

☐ 三變數函數積分的一個理解方式如: 想成超厚牛排上熱量 (卡路里) 的總和。

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

Fubini's Theorem for Triple Integrals (page 1030). *If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then*

$$\iiint_B f(x, y, z)\, \mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

Example 2 (page 1030). Evaluate the triple integral $\iiint_B xyz^2\, \mathrm{d}V$, where $B = [0, 1] \times [-1, 2] \times [0, 3]$.

Solution. Direct computation gives

$$\iiint_B xyz^2\, \mathrm{d}V = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, \mathrm{d}y\, \mathrm{d}z = \int_0^3 \frac{3z^2}{4}\, \mathrm{d}z = \frac{27}{4}.$$

Now we define the *triple integral over a general bounded region* E (一般有界區域的三重積分) in three dimensional space (a solid). We enclosed E in a box B, and then define $F(x, y, z)$ on B that it agrees with $f(x, y, z)$ on E but is 0 in B that outside E. By definition,

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \iiint_B F(x, y, z)\, \mathrm{d}V.$$

This integral exists if $f(x, y, z)$ is continuous and the boundary of E is "reasonably smooth."

Definition 3 (page 1031)**.**

(a) Region E is called *type* x if $E = \{(x, y, z)|(x, y) \in D, x_1(y, z) \leq x \leq x_2(y, z)\}$, where D is the projection of E onto the yz-plane.

(b) Region E is called *type* y if $E = \{(x, y, z)|(x, y) \in D, y_1(x, z) \leq y \leq y_2(x, z)\}$, where D is the projection of E onto the xz-plane.

(c) Region E is called *type* z if $E = \{(x, y, z)|(x, y) \in D, z_1(x, y) \leq z \leq z_2(x, y)\}$, where D is the projection of E onto the xy-plane.

Now we will change triple integrals to iterated integrals.

■ If E is *type* z and D is *type I*, then

$$\iiint_E f(x, y, z)\,\mathrm{d}V = \int_{x=a}^{x=b} \int_{y=y_1(x)}^{y=y_2(x)} \int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

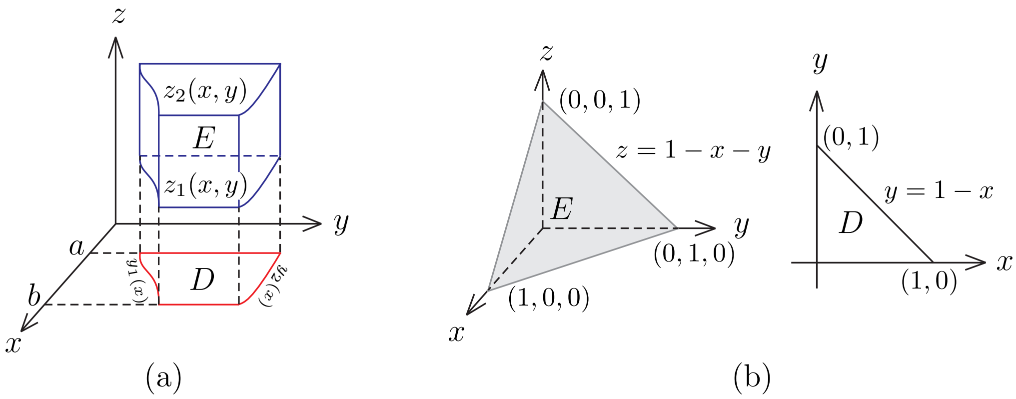

Figure 1: (a) Triple Integrals. (b) Solid E in **Example 4**.

Example 4 (page 1032)**.** Evaluate $\iiint_E z\,\mathrm{d}V$, where E is the solid tetrahedron bounded by the four planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$.

Solution.

Example 5 (page 1032)**.** Evaluate $\iiint_E \sqrt{x^2+z^2}\,\mathrm{d}V$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Solution.

Example 6 (page 1034)**.** Rewrite the iterated integral $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ in a different order, integrating first with respect to x, then z, and then y.

Solution.

Applications of Triple Integrals, page 1034

If $f(x,y,z) \equiv 1$, then the triple integral represents the volume: $V(E) = \displaystyle\iiint_E 1\,\mathrm{d}V.$

Example 7 (page 1035)**.** Use a triple integral to find the volume of the tetrahedron T bounded by the plane $x + 2y + z = 2, x = 2y, x = 0$, and $z = 0$.

Solution.

Definition 8 (page 1035–1036)**.**

(a) If the density function of a solid object that occupies the region E is $\rho(x, y, z)$ (in units of mass per unit volume), then its *mass* (質量) is

$$m = \iiint_E \rho(x, y, z)\,\mathrm{d}V.$$

(b) The *moments* (矩) of region E about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x, y, z)\,\mathrm{d}V, \qquad M_{xz} = \iiint_E y\rho(x, y, z)\,\mathrm{d}V,$$
$$M_{xy} = \iiint_E z\rho(x, y, z)\,\mathrm{d}V.$$

(c) The *center of mass* (質心) is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}.$$

(d) The *moments of inertia* (轉動慣量) about the three coordinate axes are

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\,\mathrm{d}V, \quad I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\,\mathrm{d}V,$$
$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\,\mathrm{d}V.$$

Definition 9 (page 1036)**.** The total *electric charge* on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\,\mathrm{d}V.$$

Example 10 (page 1036)**.** Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$, and the planes $x = z, z = 0$, and $x = 1$.

Solution.

Example (TA) 11. Figure 2 shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}x.$$

Rewrite this integral as an equivalent iterated integral in the five other orders.

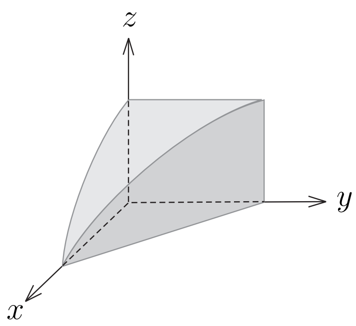

Figure 2: Integration of $f(x, y, z)$ on the solid.

Solution.

Appendix

- If E is *type z* and D is *type II*, then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_{y=c}^{y=d}\int_{x=x_1(y)}^{x=x_2(y)}\int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

- If E is *type y* and D is *type I* ($z = z_i(x)$), then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_{x=a}^{x=b}\int_{z=z_1(x)}^{z=z_2(x)}\int_{y=y_1(x,z)}^{y=t_2(x,z)} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x.$$

- If E is *type y* and D is *type II* ($x = x_i(z)$), then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_{z=r}^{z=s}\int_{x=x_1(z)}^{x=x_2(z)}\int_{y=y_1(x,z)}^{y=y_2(x,z)} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z.$$

- If E is *type x* and D is *type I* ($z = z_i(y)$), then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_{y=c}^{y=d}\int_{z=z_1(y)}^{z=z_2(y)}\int_{x=x_1(y,z)}^{x=x_2(y,z)} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y.$$

- If E is *type x* and D is *type II* ($y = y_i(z)$), then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_{z=r}^{z=s}\int_{y=y_1(z)}^{y=y_2(z)}\int_{x=x_1(y,z)}^{x=x_2(y,z)} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Exercise (page 1038)**.** Use a triple integral to find the volume of the solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$.

Exercise (page 1038)**.** Write five other iterated integrals that are equal to the iterated integral $\int_0^1\int_y^1\int_0^y f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$.

Exercise. The region E is bounded by $z = (x^2+y^2)^{\frac{1}{2}}, x^2+y^2 = 9$, and $z = 0$. Suppose the density function $\rho(x,y,z) = z$. Find the mass of E and the center of mass $(\bar{x},\bar{y},\bar{z})$.

Exercise. Find the total electric charge over the region

$$R = \{(x,y) | -1 \le x+y \le 1, -1 \le x-y \le 1\}$$

with charge density (per unit area) $\sigma(x,y) = |x| + |y|$.

Exercise. Evaluate the triple integral $\int_0^2\int_0^{4-x^2}\int_0^x \frac{\sin 2z}{4-z}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$.

Exercise. Evaluate $\int_0^1\int_0^{1-x}\int_y^1 \frac{\sin(\pi z)}{z(2-z)}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.