15.5 Surface Area, page 1026

Goal: Find the formula of the surface area of the graph of $f(x, y)$.

Let S be a surface with equation $z = f(x, y)$, where $f(x, y)$ has continuous partial derivatives. We assume that $f(x, y) \ge 0$ and the domain D of f is a rectangle. The idea is to approximate the surface area by the "tangent plane areas."

(1) Divide D into small rectangles R_{ij} with area $\Delta A = \Delta x \Delta y$.

(2) If we choose (x_i, y_i), the corner of R_{ij} closest to the origin, as a sample point, then the tangent plane to S at $P_{ij} = (x_i, y_j, f(x_i, y_j))$ is an approximation to S near P_{ij}. The area ΔT_{ij} of the part of this tangent plane that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of the part of S that lies directly above R_{ij}.

$$\begin{aligned}\Delta T_{ij} &= |\mathbf{u}_i \times \mathbf{v}_j| = |(\Delta x\,\mathbf{i} + f_x(x_i, y_i)\Delta x\,\mathbf{k}) \times (\Delta y\,\mathbf{j} + f_y(x_i, y_j)\Delta y\,\mathbf{k})| \\ &= |-f_x(x_i, y_i)\Delta x\Delta y\,\mathbf{i} - f_y(x_i, y_i)\Delta x\Delta y\,\mathbf{j} + \Delta x\Delta y\,\mathbf{k}| \\ &= \sqrt{1 + (f_x(x_i, y_i))^2 + (f_y(x_i, y_i))^2}\,\Delta A.\end{aligned}$$

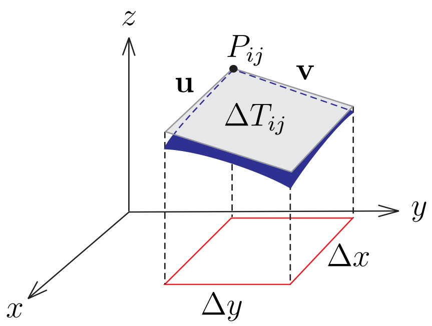

Figure 1: The area of a parallelogram $\Delta T_{ij} = |\mathbf{u}_i \times \mathbf{v}_j|$.

(3) The sum $\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta T_{ij}$ is an approximation to the total area of S.

(4) We define the *surface area* (曲面面積) of S to be

$$\text{Area}(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \sqrt{1 + (f_x(x_i, y_i))^2 + (f_y(x_i, y_i))^2}\,\Delta A.$$

Theorem 1 (page 1027). *The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is*

$$\text{Area}(S) = \iint_D \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2}\,\mathrm{d}A.$$

☐ 類比於平面曲線 $(x, y = f(x)), a \le x \le b$ 的弧長公式: $L = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x$.

Example 2 (page 1027)**.** Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Solution.

Example 3 (page 1028)**.** Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies within the cylinder $x^2 + y^2 = 2x$.

Solution.

Example (TA) 4. Find the area of the surface $z = \frac{2}{3}(x^{\frac{3}{2}} + y^{\frac{3}{2}}), 0 \le x, y \le 1$.

Solution.