15.4 Applications of Double Integrals, page 1016

In this section we explore physical applications such as computing mass, electric charge, center of mass, and moment of inertia. We will apply double integrals to probability density functions of two random variables as well.

Density and Mass, page 1016

Consider a lamina with variable density. Suppose the lamina occupies a region D of the xy-plane and its *density* (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D, then the total mass is

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x, y)\,\mathrm{d}A.$$

If an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ in D, then the total charge Q is

$$Q = \iint_D \sigma(x, y)\,\mathrm{d}A.$$

Example 1 (page 1017)**.** Charge is distributed over the region D bounded by $x = 1, y = 1$, and $x + y = 1$. The charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.

Solution.

Moments and Centers of Mass, page 1017

Suppose that the lamina occupies a region D and has density function $\rho(x, y)$. The *moment* of the entire lamina *about the y-axis* is

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^*\rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x\rho(x, y)\,\mathrm{d}A.$$

Similarly, the *moment about the x-axis* is

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^*\rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y\rho(x, y)\,\mathrm{d}A.$$

The *center of mass* $(\bar{x}, \bar{y})$ of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x, y)\, \mathrm{d}A \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x, y)\, \mathrm{d}A,$$

where the mass m is given by $m = \iint_D \rho(x, y)\, \mathrm{d}A$.

Example 2 (page 1018)**.** The density at any point on a semicircular lamina with radius R is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

Solution.

Moment of Inertia, page 1019

The *moment of inertia* (also called the *second moment*) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. The *moment of inertia* of the lamina *about the x-axis* is

$$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D y^2 \rho(x, y)\, \mathrm{d}A.$$

Similarly the *moment of inertia about the y-axis* is

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D x^2 \rho(x, y)\, \mathrm{d}A.$$

It is also of interest to consider the *moment of inertia about the origin*, also called the *polar moment of inertia*:

$$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left((x_{ij}^*)^2 + (y_{ij}^*)^2\right) \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D (x^2 + y^2) \rho(x, y)\, \mathrm{d}A.$$

Note that $I_0 = I_x + I_y$. The *radius of gyration* (迴轉半徑) of a lamina about an axis is the number R such that $mR^2 = I$, where m is the mass of the lamina and I is the moment of inertia about the given axis. In particular, the radius of gyration $\bar{\bar{y}}$ with respect to the x-axis and the radius of gyration $\bar{\bar{x}}$ with respect to the y-axis are given by the equation $m\bar{\bar{y}}^2 = I_x$ and $m\bar{\bar{x}}^2 = I_y$.

Example 3 (page 1020–1021)**.**

(a) Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin, and radius R.

(b) Find the radius of gyration about the x-axis of the disk D.

Solution.