15.3 Double Integrals in Polar Coordinates, page 1010

<u>Goal</u>: We want to evaluate a double integral $\iint_R f(x,y)\,\mathrm{d}A$, where R is easily described using polar coordinates.

Recall that relations between Cartesian coordinates and polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x}. \end{cases}$$

Definition 1 (page 1010)**.** Define the *polar rectangle* (極坐標的範圍)

$$R = \{(r,\theta) | a \le r \le b, \alpha \le \theta \le \beta\}.$$

Here we use the definition of double integral to find the formula of double integrals in polar rectangles.

(1) Divide the polar rectangle into small polar rectangles:

- Dividing $[a,b]$ into m subinterval $[r_{i-1}, r_i]$ with width $\Delta r = \frac{b-a}{m}$.
- Dividing $[\alpha,\beta]$ into n subinterval $[\theta_{i-1}, \theta_i]$ with width $\Delta\theta = \frac{\beta-\alpha}{n}$.
- We get small polar rectangles: $R_{ij} = \{(r,\theta) | r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$.

(2) Choose the "center" of the polar subrectangle (r_i^*, θ_j^*) as the *sample point* in each R_{ij}, where

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i), \qquad \theta_j = \frac{1}{2}(\theta_{j-1} + \theta_j).$$

We compute the area of R_{ij}:

$$\Delta A_{ij} = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^* \Delta r \Delta\theta.$$

(3) We get the double Riemann sum in polar rectangles:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*) r_i^* \Delta r \Delta\theta.$$

(4) When $m, n \to \infty$, we get

$$\begin{aligned} \iint_R f(x,y)\,\mathrm{d}A &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*) r_i^* \Delta r \Delta\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r \,\mathrm{d}r \mathrm{d}\theta. \end{aligned}$$

Change to Polar Coordinates in a Double Integral (page 1012). *If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then*

$$\iint_R f(x, y)\,\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

☐ 極坐標的面元是 $r\,\mathrm{d}r\,\mathrm{d}\theta$; 幾何上可想成微小區域面積 $\mathrm{d}r\,r\mathrm{d}\theta$: 徑向長乘上圓弧長。

☐ $r\,\mathrm{d}r\,\mathrm{d}\theta$ 前面的 r 稱爲 Jacobian: $\frac{\partial(x,y)}{\partial(r,\theta)}$。

Example 2 (page 1012). Evaluate $\iint_R (3x + 4y^2)\,\mathrm{d}A$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution.

Example 3 (page 1012). Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Solution.

Theorem 4 (page 1013)**.** *If $f(x,y)$ is continuous on a polar region of the form $D = \{(r,\theta)|h_1(\theta) \le r \le h_2(\theta), \alpha \le \theta \le \beta\}$, then*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_\alpha^\beta \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

□ 先對 r 方向積分, 上限爲 $r = h_2(\theta)$, 下限爲 $r = h_1(\theta)$, 皆爲 θ 的函數。

Example 5 (page 1013)**.** Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution.

Example 6 (page 1014)**.** Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Solution.

Exercise. Find the integral $\iint_R |y|\,\mathrm{d}A$, where D is the region enclosed by $r = 4 + 3\cos\theta$.

Exercise. Find the integral $\displaystyle\iint_R \frac{1}{(1+x^2+y^2)^2}\,\mathrm{d}A$, where R is the region enclosed by the Lemniscate $r^2 = \cos 2\theta$.

Exercise. Find the volume bounded by the sphere $x^2+y^2+z^2=4$ and the cylinder $(x-1)^2+y^2=1$.

Exercise. Evaluate the integral $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}y\,\mathrm{d}x$

Exercise. Find the integral $\displaystyle\int_{0}^{\frac{1}{2}}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} \ln(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y.$

Example (TA) 7 (page 1015)**.** Find the integral $\displaystyle\iint_{\mathbb{R}^2} \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}A.$

Solution.

Appendix

The area form (or volume form) has the following structure: Since

$$\begin{cases} \mathrm{d}x = \mathrm{d}r\cos\theta - r\sin\theta\,\mathrm{d}\theta \\ \mathrm{d}y = \mathrm{d}r\sin\theta + r\cos\theta\,\mathrm{d}\theta \end{cases} \Rightarrow \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \mathrm{d}r \\ \mathrm{d}\theta \end{bmatrix}.$$

We compute the determinant of the matrix, called *Jacobian*:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Thus the area form is $\mathrm{d}x\wedge\mathrm{d}y = r\,\mathrm{d}r\wedge\mathrm{d}\theta$.