15.2 Double Integrals over General Regions, page 1001

Goal: We will learn how to integrate a function $f(x, y)$ over a bounded region D.

Define a new function $F(x, y)$ with a rectangular region $R \supseteq D$ by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in D.} \end{cases} \tag{1}$$

Figure 1: Double integral of f over D.

Definition 1 (page 1001)**.** If $F(x, y)$ is integrable over R, then we define the *double integral of* $f(x, y)$ *over* D (區域 D 上函數 $f(x, y)$ 的重積分) by

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A, \quad \text{where } F \text{ is given by (1).}$$

Definition 2 (page 1002)**.**

(1) A plane region D is said to be of *type I* if it lies between the graphs of two continuous functions of x, that is, $D = \{(x, y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$, where $g_1(x)$ and $g_2(x)$ are continuous on $[a, b]$.

(2) A plane region D is said to be of *type II* if it lies between the graphs of two continuous functions of y, that is, $D = \{(x, y) | h_1(y) \le x \le h_2(y), c \le y \le d\}$, where $h_1(y)$ and $h_2(y)$ are continuous on $[c, d]$.

Figure 2: Type I and Type II region.

Theorem 3 (page 1002–1003)**.**

(a) *If $f(x,y)$ is continuous on a type I region D, then*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \left(\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

(b) *If $f(x,y)$ is continuous on a type II region D, then*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_c^d \left(\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

Proof of (a). We choose a rectangle $R = [a,b] \times [c,d]$, where c and d are constants satisfy $c \leq \min_{x\in[a,b]} g_1(x)$ and $d \geq \max_{x\in[a,b]} g_2(x)$. Let $F(x,y)$ be the function given by (1). By Fubini's Theorem, we have

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A = \int_a^b \left(\int_c^d F(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

For fixed $x \in [a,b]$, since $F(x,y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$, the lower limit can be replaced by $g_1(x)$, and the upper limit can replaced by $g_2(x)$. Therefore,

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} F(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

because $F(x,y) = f(x,y)$ when $g_1(x) \leq y \leq g_2(y)$. □

- ☐ 非矩形區域上函數的積分, 上下限範圍改成「函數」。
- ☐ 若是先對 y 積分, 上、下限當然是 y 的上下限, 所以是 $y = g_2(x)$ 與 $y = g_1(x)$。
- ☐ 學會畫圖 (定義域); 熟練區域變換。

Example 4 (page 1003)**.** Evaluate $\iint_D (x+2y)\,\mathrm{d}A$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = x^2 + 1$.

Example 5. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = x$ and the parabola $y = x^2$.

Solution.

Solution 2.

Example 6. Evaluate $\iint_D xy \, \mathrm{d}A$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = x + 1$.

Solution.

Example 7 (page 449, 1008)**.** Find the volume common to two circular cylinders, each with radius r, if the axes of the cylinders intersect at right angles.

Solution.

Solution 2.

Example 8 (page 1006)**.** Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,\mathrm{d}y\,\mathrm{d}x$.

Solution.

Properties of Double Integrals, page 1006

We assume that all of the integrals exist.

(a) $\displaystyle\iint_D (f(x,y)+g(x,y))\,\mathrm{d}A = \iint_D f(x,y)\,\mathrm{d}A + \iint_D g(x,y)\,\mathrm{d}A.$

(b) $\displaystyle\iint_D cf(x,y)\,\mathrm{d}A = c\iint_D f(x,y)\,\mathrm{d}A.$

(c) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in D$, then $\displaystyle\iint_D f(x,y)\,\mathrm{d}A \geq \iint_D g(x,y)\,\mathrm{d}A.$

(d) If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except on boundaries, then

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A.$$

This property can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II.

Figure 3: D is neither type I nor type II. D_1 is type I, D_2 is type II.

(e) $\displaystyle\iint_D 1\,\mathrm{d}A = \operatorname{Area}(D).$

(f) If $m \leq f(x,y) \leq M$ for all (x,y) in D, then

$$m\operatorname{Area}(D) \leq \iint_D f(x,y)\,\mathrm{d}A \leq M\operatorname{Area}(D).$$

Exercise. Find the integral $\displaystyle\int_0^8\int_{\sqrt[3]{y}}^2 \sqrt{1+x^4}\,\mathrm{d}x\,\mathrm{d}y.$

Exercise. Evaluate the integral $\displaystyle\int_0^{\sqrt{\pi}}\int_{\frac{x}{2}}^{\frac{\sqrt{\pi}}{2}} \sin(y^2)\,\mathrm{d}y\,\mathrm{d}x.$

Exercise. Evaluate the integral $\displaystyle\int_0^4\int_{\frac{y}{2}}^2 \mathrm{e}^{x^2}\,\mathrm{d}x\,\mathrm{d}y.$

Exercise (page 1008)**.** Find the volume of the solid bounded by the cylinder $y^2+z^2=4$ and the planes $x=2y$, $x=0$, $z=0$ in the first octant.

Example (TA) 9. Find the integral $\iint_R |y - x^2| \, \mathrm{d}A$, $R = [-1, 1] \times [0, 2]$.

Solution.