Chapter 15 Multiple Integrals

15.1 Double Integrals over Rectangles, page 988

Review of the Definite Integral, page 988

Suppose that $f(x)$ is defined for $a \leq x \leq b$.

(1) Divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$.

(2) In each subinterval $[x_{i-1}, x_i]$, choose a sample point $x_i^* \in [x_{i-1}, x_i]$.

(3) Define the Riemann sum (黎曼和) $= \sum_{i=1}^{n} f(x_i^*)\Delta x$.

(4) Define the *definite integral* (定積分) of $f(x)$ from a to b by

$$\int_a^b f(x)\,\mathrm{d}x \overset{\text{def.}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

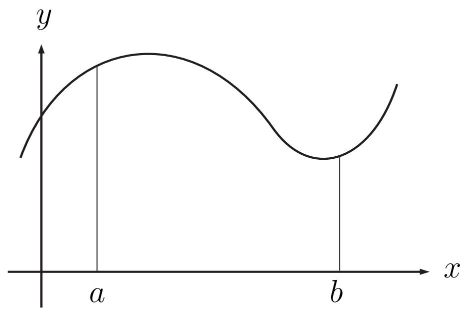

Figure 1: Definition of a definite integral.

☐ 幾何意義:「有向」面積; 函數在 x-軸的上方或下方; 積分範圍從左至右或從右至左。

Volumes and Double Integrals, page 988

Similarly, we consider a function $f(x, y)$ defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\},$$

and we first suppose that $f(x, y) \geq 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is

$$S = \{(x, y, z) \in \mathbb{R}^3 | 0 \leq z \leq f(x, y), (x, y) \in R\},$$

The goal is to find the volume of S.

(1) Divide the rectangle R into subrectangles:

- Divide $[a,b]$ into m subinterval $[x_{i-1}, x_i]$ with width $\Delta x = \frac{b-a}{m}$.
- Divide $[c,d]$ into n subinterval $[y_{i-1}, y_i]$ with width $\Delta y = \frac{d-c}{n}$.
- We form the subrectangles:
  $R_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] = \{(x,y) | x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$.
  Each R_{ij} with area $\Delta A = \Delta x \Delta y$.

(2) Choose a *sample point* (x_{ij}^*, y_{ij}^*) (樣本點) in each R_{ij}.

(3) We get an approximation to the total volume of S by *double Riemann sum* (二重黎曼和):

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

(4) Define the *volume* (體積) of the solid S by

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

Definition 1 (page 990)**.** The *double integral* (重積分, 二重積分) of $f(x,y)$ over the rectangle R is

$$\iint_R f(x,y)\,\mathrm{d}A \stackrel{\text{def.}}{=} \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

if this limit exists.

☐ 幾何意義:「有向」體積; 函數在 xy-平面的上方或下方。

☐ 寫成 $\mathrm{d}A$ 這個符號時, 總是代表「正的面元」。

☐ 使用記號 $\mathrm{d}x \wedge \mathrm{d}y$ (右手定則) 代表「有向面元」。($\mathrm{d}y \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}y$)

Iterated Integrals, page 993

Goal: Compute the double integrals by iterated integrals.
Recall that it is usually difficult to evaluate single integrals directly from the definition of an integral. The evaluation of double integrals from the definition is even more difficult. In this section, we will see how to express a double integral as an iterated integral, which can be evaluated by calculating two single integrals.

Suppose that $f(x,y)$ is integrable on the rectangle $R = [a,b] \times [c,d]$.

Definition 2 (page 993)**.** Define the *partial integration of $f(x,y)$ with respect to y*, denoted by $\int_c^d f(x,y)\,\mathrm{d}y$ to mean that x is fixed and $f(x,y)$ is integrated with respect to y from $y=c$ to $y=d$.

After partial integration, $\int_c^d f(x,y)\,\mathrm{d}y$ depends on x, so we denote it by $A(x)$.

Definition 3 (page 993)**.** If we integrate $A(x)=\int_c^d f(x,y)\,\mathrm{d}y$ with respect to x from $x=a$ to $x=b$, we get the *iterated integral* (先對 y 後對 x 的二次積分):

$$\int_a^b A(x)\,\mathrm{d}x = \int_a^b \left(\int_c^d f(x,y)\,\mathrm{d}y\right)\mathrm{d}x = \int_a^b\int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

☐ 雖然有時候大括號會省略, 但還是建議添加。

Similarly, the iterated integral (先對 x 後對 y 的二次積分)

$$\int_c^d\int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_a^b f(x,y)\,\mathrm{d}x\right)\mathrm{d}y = \int_c^d B(y)\,\mathrm{d}y.$$

means that we first integrate with respect to x (fixed y) from $x=a$ to $x=b$ and then integrate the resulting function $B(y)=\int_a^b f(x,y)\,\mathrm{d}x$ from $y=c$ to $y=d$.

Fubini's Theorem (page 994)**.** *If $f(x,y)$ is continuous on the rectangles $R=\{(x,y)|a\le x\le b, c\le y\le d\}$, then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b \left(\int_c^d f(x,y)\,\mathrm{d}y\right)\mathrm{d}x = \int_c^d \left(\int_a^b f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

In general, this is true if we assume that $f(x,y)$ is bounded on R, $f(x,y)$ is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 4 (page 995)**.** Evaluate $\iint_R y\sin(xy)\,\mathrm{d}A$, where $R=[1,2]\times[0,\pi]$.

Solution. If we first integrate with respect to x, we get

$$\iint_R y\sin(xy)\,\mathrm{d}A =$$

Solution 2. If we reverse the order of integration, we get

$$\iint_R y\sin(xy)\,\mathrm{d}A =$$

We use ____________________ and get

So

☐ 有時候只有一種方式「積得出來」, 所以積分「先後順序的轉換」要熟練並會巧妙變換。

Example 5 (page 996)**.** Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the plane $x = 2$ and $y = 2$, and the three coordinate planes.

Solution.

In the special case, where $f(x, y) = g(x)h(y)$ is the product of a function of x only and a function of y only, by Fubini's Theorem, we get

Example 6 (page 996)**.** If $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$, then

$$\iint_R \sin x \cos y \, \mathrm{d}A =$$