14.7 Maximum and Minimum Values, page 959

Definition 1 (page 960)**.** A function of two variables has a *local maximum* (局部極大值處) at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ when (x, y) is near (x_0, y_0). (This means that $f(x, y) \leq f(x_0, y_0)$ for all points (x, y) in some disk with center (x_0, y_0).) The number $f(x_0, y_0)$ is called a *local maximum value* (局部極大值). If $f(x, y) \geq f(x_0, y_0)$ when (x, y) is near (x_0, y_0), then f has a *local minimum* (局部極小值處) at (x_0, y_0) and $f(x_0, y_0)$ is a *local minimum value* (局部極小值).

Definition 2 (page 960)**.** If the inequalities in Definition 1 hold for *all* points (x, y) in the domain of f, then f has an *absolute maximum* (最大值) or *absolute minimum* (最小值) at (x_0, y_0).

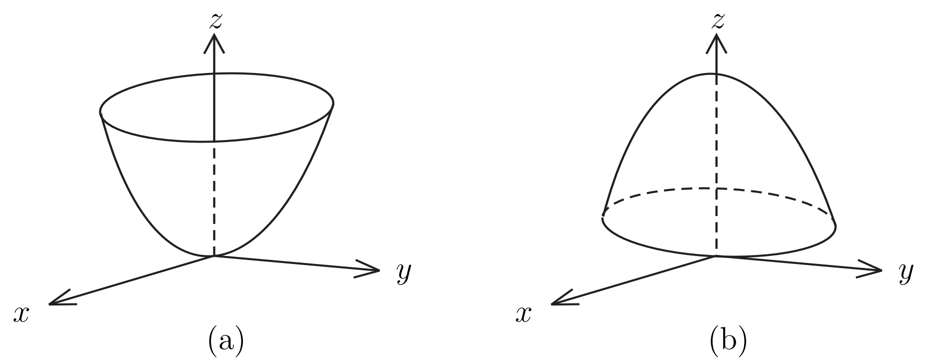

Figure 1: (a) Local and absolute minimum. (b) Local and absolute maximum.

Theorem 3 (page 960)**.** *If f has a local maximum or minimum at (x_0, y_0) and the first-order partial derivatives of f exist there, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.*

Proof. Let $g(x) = f(x, y_0)$. If f has a local maximum (or minimum) at (x_0, y_0), then $g(x)$ has a local maximum (or minimum) at x_0, so by Fermat's Theorem, we get $g'(x_0) = 0 = f_x(x_0, y_0)$. Similarly, by applying Fermat's Theorem to the function $\tilde{g}(y) = g(x_0, y_0)$, we obtain $g'(y_0) = 0 = f_y(x_0, y_0)$. □

□ 若函數在 (x_0, y_0) 是局部極值, 則 $\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0)) = (0, 0) = \mathbf{0}$。

Definition 4 (page 960)**.** A point (x_0, y_0) is called a *critical point* (臨界點) or *stationary point* (平穩點、駐點) of f if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or if one of these partial derivatives does not exist.

□ 臨界點除了滿足 $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ 的點外, 還包括所有偏導數不存在的點。

□ 臨界點只是函數達到極值的必要條件, 非充分條件。

Example 5 (page 960)**.** Consider $f(x, y) = x^2 + y^2 - 2x - 6y + 14$, then

$$f_x(x, y) = 2x - 2 \qquad f_y(x, y) = 2y - 6.$$

These partial derivative are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. Since $f(x, y) = 4 + (x - 1)^2 + (y - 3)^3 \geq 4$ for all x and y, $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f.

The graph of f is the ____________________ with vertex $(1, 3, 4)$.

Example 6 (page 960)**.** Consider the function $f(x, y) = y^2 - x^2$. Since $f_x = -2x$ and $f_y = 2y$, the only critical point is ______. For points on the x-axis ($x \neq 0$) we have $f(x, 0) = -x^2 < 0$ and for points on the y-axis ($y \neq 0$) we have $f(x, 0) = y^2 > 0$. Thus every disk with center $(0, 0)$ contains points where f takes positive values and negative values. Therefore, f has no extreme value.

The graph of f is the ______________________.

Definition 7 (page 961)**.** The graph of $z = y^2 - x^2$ has a horizontal tangent plane $z = 0$ at the origin. $f(0, 0) = 0$ is a maximum in the direction of x-axis but a minimum in the direction of the y-axis. Near the origin the graph has the shape of a saddle and so $(0, 0)$ is called a *saddle point* (鞍點) of f.

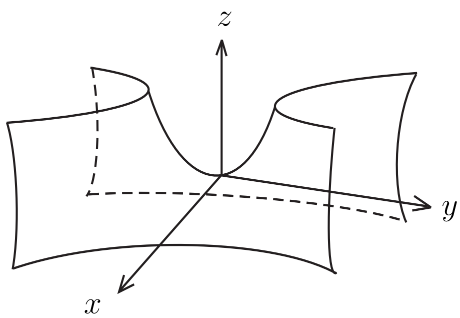

Figure 2: $(0, 0)$ is a saddle point of $f(x, y) = y^2 - x^2$.

For a function of one variable $f(x)$, we use second derivative of $f(x)$ to detect the critical points are local maximum or local minimum. Here we will introduce the Second Derivative Test for functions of two variables to investigate the properties of critical points.

Definition 8. The *Hessian matrix* or *Hessian* (赫氏矩陣) of $f(x, y)$ at (x_0, y_0) is

$$\text{Hess}\,(f)(x_0, y_0) = \begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}.$$

Second Derivative Test (page 961). *Suppose the second partial derivatives of f are continuous on a disk with center (x_0, y_0), and suppose that $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ (that is, (x_0, y_0) is a critical point of f). Let*

$$D(x_0, y_0) = \det(\text{Hess}(f)(x_0, y_0)) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.$$

(a) *If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is a local minimum.*

(b) *If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a local maximum.*

(c) *If $D(x_0, y_0) < 0$, then $f(x_0, y_0)$ is not a local maximum or minimum.*

☐ 情況 (c) 是鞍點。

☐ 若 $D(x_0, y_0) = 0$, 三種情況 (極大、極小、鞍點) 都有可能發生; 必須用別的方法判斷。

Remark 9.

(1) 對稱矩陣可以對角化, 所以 $\text{Hess}(f) = PDP^{-1}$, 其中 D 是對角化矩陣; P 是坐標變換矩陣。

(2) 因為 $\det(AB) = \det(BA)$, 所以行列式在坐標變換下不變; 即 $\det(\text{Hess}(f)) = \det(PDP^{-1}) = \det(PP^{-1}D) = \det(ID) = \det(D)$。

(3) $f_{xx} > 0$ 沿 x 方向凹口向上, 則對角矩陣其中一個值 (特徵值、固有值) 為正; $f_{xx} < 0$ 沿 x 方向凹口向下, 則對角矩陣其中一個值 (特徵值、固有值) 為負。

Example 10. Find the extreme value (local maximum and minimum values and saddle points) of the function $f(x, y) = 2x^3 - 4xy + 3y^2$.

Solution.

Absolute Maximum and Minimum Values, page 965

Recall that for one variable function $f(x)$, the Extreme Value Theorem says that if f is continuous on a *closed* interval $[a, b]$, then f has an absolute maximum value and an absolute minimum value. Absolute maximum and absolute minimum points are happened at the critical points or endpoints.

We will introduce the Extreme Value Theorem of two variables in this section.

Definition 11 (page 965)**.**

(a) A *boundary point* (邊界點) of a set $D \subset \mathbb{R}^2$ is a point (x_0, y_0) such that every disk with center (x_0, y_0) contains points in D and also points not in D.

(b) A *closed set* (閉集) D in $\mathbb{R}^2$ is one that contains all its boundary points.

(c) A *bounded set* (有界集) D in $\mathbb{R}^2$ is one that is contained within some disk.

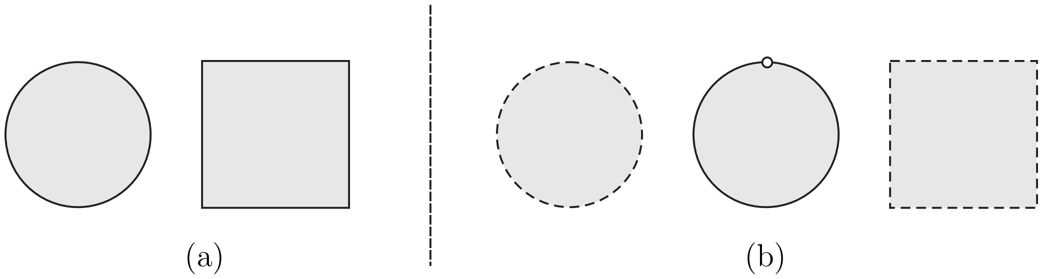

Figure 3: (a) Closed sets. (b) Sets that are not closed.

Extreme Value Theorem for Functions of Two Variables (page 965)**.** *If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.*

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

(a) Find the values of f at the critical points of f in D.

(b) Find the extreme values of f on the boundary of D.

(c) The largest of the values from step 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

☐ 找「有界區域」的函數極值, 除了臨界點外, 邊界點也要納入考慮。

Example 12. Find the extreme values of $f(x, y) = x^2 + xy + y^2 - 4x + 3y$ in the region bounded by $x = 0, y = 0$, and $x + y = 4$.

Solution.

Example 13. Find the absolute maximum and minimum values of $f(x, y) = 4x + 6y - x^2 - y^2$ in the region $x^2 + y^2 \leq 1$.

Solution.

Appendix, page 967

Proof of the Second Derivative Test. We compute the second-order directional derivative of f in the direction of $\mathbf{u} = (h, k)$. The first-order derivative is

$$D_{\mathbf{u}}f = f_x h + f_y k.$$

Apply this theorem a second time, we have

$$\begin{aligned} D^2_{\mathbf{u}}f = D_{\mathbf{u}}(D_{\mathbf{u}}f) = \nabla(D_{\mathbf{u}}f) \cdot \mathbf{u} &= \left(\frac{\partial}{\partial x}(f_x h + f_y k), \frac{\partial}{\partial y}(f_x h + f_y k)\right) \cdot (h, k) \\ &= (f_{xx}h + f_{yx}k)h + (f_{xy}h + f_{yy})k = f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \\ &= f_{xx}\left(\frac{f_{xx}h + f_{xy}k}{f_{xx}}\right)^2 + \frac{k^2}{f_{xx}}(f_{xx}f_{yy} - f_{xy}^2). \end{aligned}$$

Remark that $D(x_0, y_0) = \det(\text{Hess}(f)(x_0, y_0)) = \left(f_{xx}f_{yy} - f_{xy}^2\right)\big|_{(x_0,y_0)}$.

(a) If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then $D^2_{\mathbf{u}}f(x_0, y_0) \geq 0$ for any direction $\mathbf{u}$. If $D^2_{\mathbf{u}}f(x_0, y_0) = 0$ for some direction $\mathbf{u}$, then $k = 0$ and $f_{xx}h + f_{xy}k = 0$. However, it implies $(h, k) = (0, 0)$ and it contradicts to $h^2 + k^2 = 1$. Hence $f(x_0, y_0)$ is a local minimum.

(b) If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then $D^2_{\mathbf{u}}f(x_0, y_0) \leq 0$ for any direction $\mathbf{u}$. If $D^2_{\mathbf{u}}f(x_0, y_0) = 0$ for some direction $\mathbf{u}$, then $k = 0$ and $f_{xx}h + f_{xy}k = 0$. However, it implies $(h, k) = (0, 0)$ and it contradicts to $h^2 + k^2 = 1$. Hence $f(x_0, y_0)$ is a local maximum.

(c) If $D(x_0, y_0) < 0$, we will find two different directions such that the signs of the second derivatives of f are different.

- ⋆ If $f_{xx} > 0$ and $f_{yy} < 0$, then choosing $\mathbf{u} = (h, k) = (1, 0)$ implies $D^2_{\mathbf{u}}f = f_{xx} > 0$. If we choose $(h, k) = (0, 1)$, then $D^2_{\mathbf{u}}f = f_{yy} < 0$.
- ⋆ If $f_{xx} < 0$ and $f_{yy} > 0$, then choosing $\mathbf{u} = (h, k) = (1, 0)$ implies $D^2_{\mathbf{u}}f = f_{xx} < 0$. If we choose $(h, k) = (0, 1)$, then $D^2_{\mathbf{u}}f = f_{yy} > 0$.
- ⋆ If $f_{xx} > 0$ and $f_{yy} > 0$, then choosing $\mathbf{u} = (h, k) = (1, 0)$ implies $D^2_{\mathbf{u}}f = f_{xx} > 0$. If we choose $(h, k) /\!/ (f_{xy}, -f_{xx})$ such that $k \neq 0$, then $D^2_{\mathbf{u}}f = \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}}\right)k^2 < 0$.
- ⋆ If $f_{xx} < 0$ and $f_{yy} < 0$, then choosing $\mathbf{u} = (h, k) = (1, 0)$ implies $D^2_{\mathbf{u}}f = f_{xx} < 0$. If we choose $(h, k) /\!/ (f_{xy}, -f_{xx})$ such that $k \neq 0$, then $D^2_{\mathbf{u}}f = \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}}\right)k^2 > 0$.

Hence $f(x_0, y_0)$ is not a local maximum or minimum.

□