14.6 Directional Derivatives and the Gradient Vector, page 946

Directional Derivatives, page 946

Definition 1 (page 947)**.** The *directional derivative* (方向導數) of $f(x, y)$ at (x_0, y_0) in the direction of a *unit* vector $\mathbf{u} = (a, b)$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

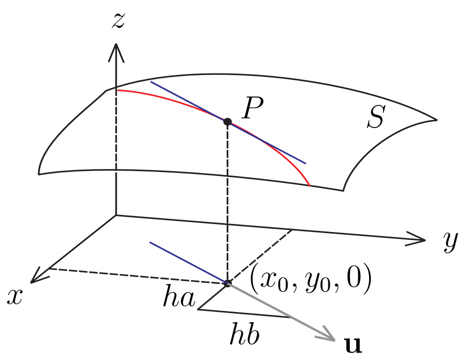

Figure 1: Directional derivative.

☐ $\mathbf{u}$ 必須是單位向量; 有時候只告知方向, 必須先把向量「單位化」後再計算方向導數。

Theorem 2 (page 948)**.** *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = (a, b)$ and*

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = (f_x(x, y), f_y(x, y)) \cdot (a, b).$$

Proof. Define $g(h) = f(x(h), y(h)) = f(x_0 + ha, y_0 + hb)$, then by the definition of a directional derivative and the Chain Rule, we have

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) = g'(0) &= \left[\frac{\partial f}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}h} + \frac{\partial f}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}h} \right] \bigg|_{h=0} \\ &= f_x(x_0, y_0)a + f_y(x_0, y_0)b = (f_x(x_0, y_0), f_y(x_0, y_0)) \cdot (a, b). \end{aligned}$$

☐

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then we can write $\mathbf{u} = (\cos\theta, \sin\theta)$ and the directional derivative becomes

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos\theta + f_y(x, y) \sin\theta = (f_x(x, y), f_y(x, y)) \cdot (\cos\theta, \sin\theta).$$

The Gradient Vector, page 949

Definition 3 (page 950)**.** If f is a function of two variables x and y, then the *gradient* (梯度) of f is the vector function ∇f or $\operatorname{grad} f$ defined by

$$\nabla f(x,y) = \operatorname{grad} f(x,y) \stackrel{\text{def.}}{=} (f_x(x,y), f_y(x,y)) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}.$$

Theorem 4 (page 950)**.** *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = (a,b)$ and*

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

☐ 方向導數爲「梯度向量」與「單位向量」內積。

☐ 函數 $f(x,y)$ 的梯度向量 $\nabla f = (f_x, f_y)$ 是在 xy-平面上。

Maximizing the Directional Derivative, page 952

Theorem 5 (page 952)**.** *Suppose f is a differentiable function of two variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(x,y)$ is $|\nabla f(x,y)|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(x,y)$.*

Proof. Since $|\mathbf{u}| = 1$, we have

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta,$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}} f$ is ∇f when $\mathbf{u}$ is the same direction as ∇f. ☐

☐ 柯西不等式 (Cauchy inequality) $\mathbf{u} \cdot \mathbf{v} \leq \|\mathbf{u}\|\|\mathbf{v}\|$。

Example 6. Let $f(x,y) = 2x^2 - xy + y^2 - 2x + y$.

(a) Find the directional derivative $D_{\mathbf{u}} f(p)$, where $p = (0,0)$ and $\mathbf{u} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

(b) Find the unit vector $\mathbf{v}$ that the directional derivative $D_{\mathbf{v}} f(p)$ is maximal.

Solution.

Functions of Three Variables, page 950

Using vector notation, we can write the directional derivative in the compact form:

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h\to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h} = \nabla f(\mathbf{x}_0) \cdot \mathbf{u}.$$

where $\mathbf{x}_0 = (x_0, y_0)$ if $n = 2$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$ if $n = 3$.

Tangent Planes to Level Surfaces, page 954

Suppose S is a level surface with equation $F(x, y, z) = k$, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P, that is, C is parameterized by $\mathbf{r}(t) = (x(t), y(t), z(t))$ and $\mathbf{r}(t_0) = (x(t_0), y(t_0), z(t_0)) = (x_0, y_0, z_0)$. Since C lies on S, we know

$$F(x(t), y(t), z(t)) = k. \tag{1}$$

If $x, y,$ and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of equation (1) as follows:

$$\frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = 0 \Rightarrow \nabla F \cdot \mathbf{r}'(t) = 0.$$

In particular, when $t = t_0$, we have $\nabla F(x_0.y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$

> The gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(0)$ to any curve C on S that passes through P.

Definition 7 (page 954)**.** If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the *tangent plane to the level surface* $F(x, y, z) = k$ at P (等位面的切平面) as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. The equation is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0. \tag{2}$$

☐ 計算等位面的切平面, 梯度即爲切平面的法向量。

Definition 8 (page 954)**.** The *normal line* (法線) to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is the gradient vector $\nabla F(x_0, y_0, z_0)$, and so its symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}. \tag{3}$$

☐ 若曲面可表示爲函數的圖形 $z = f(x, y)$, 可想成 $F(x, y, z) = z - f(x, y) = 0$。

Example 9 (page 941)**.** Find the equations of the tangent plane and normal line at $P(-2, 1, 3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

Solution.

Significance of the Gradient Vector, page 955

Consider a function of two variables $f(x, y)$.

(1) The gradient vector ∇f is orthogonal to the level curve $f(x, y) = k$.

(2) The gradient vector ∇f gives the direction of fastest increases of f.

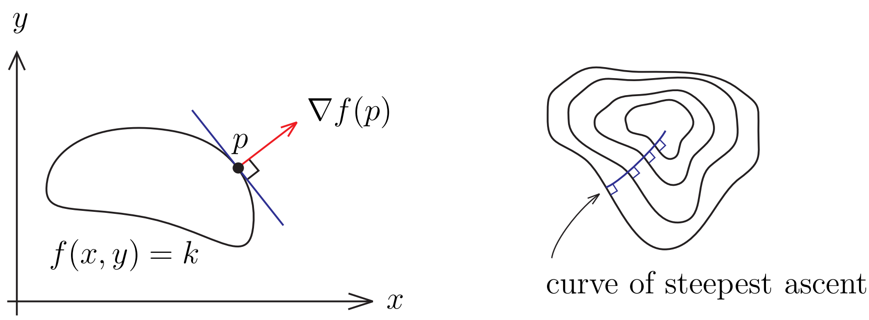

Figure 2: The gradient vector is orthogonal to the level curve.

Consider a function of three variables $F(x, y, z)$.

(1) The gradient vector ∇F is orthogonal to the level surface $F(x, y, z) = k$.

(2) The gradient vector ∇F gives the direction of fastest increases of F.

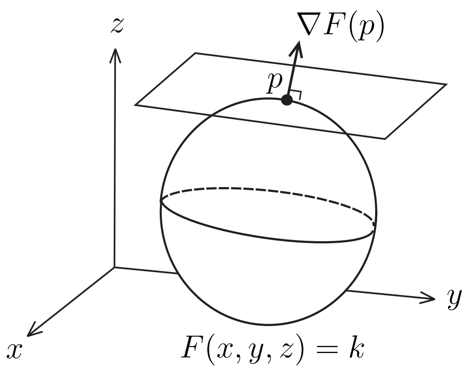

Figure 3: The gradient vector is orthogonal to the level surface.

Intersection of Two Surfaces

Suppose S_1 and S_2 are two surfaces determined by two equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$, respectively. The intersection of two surfaces is a space curve called C. Suppose that $\mathbf{r}(t)$ is a parametric equation of the space curve C and $\mathbf{r}(t_0) = P$, then $\mathbf{r}'(t_0)$ is parallel to $\nabla F(p) \times \nabla G(p)$.

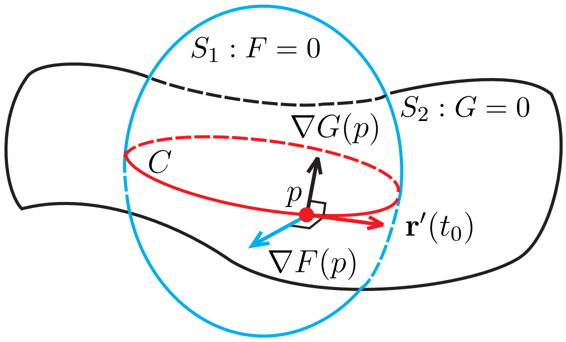

Figure 4: Intersection of two surfaces.

Example 10. Find the parametric equation of the tangent line to the curve of intersection of the surfaces $x^2 + 2y^2 + z^2 = 4$ and $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$.

Solution.