14.5 The Chain Rule, page 937

The Chain Rule, Case 1 (page 938)**.** *Suppose that $z = z(x, y)$ is a differentiable function of x and y, where $x = x(t)$ and $y = y(t)$ are both differentiable function of t. Then z is a differentiable function of t and*

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

☐ z 和 x, y 與 t 的關係式為 $z(t) = z(x(t), y(t))$。

Example 1 (page 938)**.** If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{\mathrm{d}z}{\mathrm{d}t}$ when $t = 0$.

Solution.

Example 2. Find the second derivative $\frac{\mathrm{d}^2z}{\mathrm{d}t^2}$.

Solution.

The Chain Rule, Case 2 (page 939)**.** *Suppose that $z = z(x, y)$ is a differentiable function of x and y, where $x = x(s, t)$ and $y = y(s, t)$ are differentiable functions of s and t. Then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

☐ z 與 x, y 和 x, t 的關係式為 $z(s, t) = z(x(s, t), y(s, t))$。

Case 2 of the Chain Rule contains three types of variables: s and t are *independent variables*, x and y are *intermediate variables*, and z is the *dependent variable*.

Example 3. If $z = \mathrm{e}^x \sin y$, where $x = st^2$ and $y = s^2t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution.

Example 4. Let $z = y + f(x^2 - y^2)$ and f be a differentiable function of single variable. Find $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$.

Solution.

The Chain Rule, General Version (page 940)**.** *Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$, and each x_i is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and*

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i}.$$

☐ 以上鏈鎖律爲合成函數的求導法則。

Coordinates Changes

In $\mathbb{R}^2$, denote (x, y) as the Cartesian coordinates and (r, θ) as the polar coordinates. We know relations between these coordinates are

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}.$$

So we know $x = x(r, \theta), y = y(r, \theta)$ and $r = r(x, y), \theta = \theta(x, y)$, and hence

$$x = x(r(x, y), \theta(x, y)) \quad \text{and} \quad y = y(r(x, y), \theta(x, y)).$$

Since x and y are independent variables, we have

$$\left\{ \qquad \qquad \qquad \Rightarrow \right.$$

So partial derivatives of coordinates changes form inverse matrices. Now we check this relation by computing partial derivatives directly.

☐ 單變數的情形 $\frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}y} = 1$; 多變數的情形: 矩陣相乘爲單位矩陣。

☐ 有時候坐標變換關係式很複雜 (隱函數), 不易求偏導數, 計算反矩陣比較快。

Example 5. Consider $z = f(x, y)$, where all the second partial derivatives of f are continuous. Let $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Express $\frac{\partial^2 r}{\partial x^2}$ and $\frac{\partial^2 \theta}{\partial x^2}$ in terms of r and θ.

(b) Express f_{xx} in terms of $r, \theta, f_r, f_\theta, f_{rr}, f_{r\theta}$, and $f_{\theta\theta}$.

Solution.

Implicit Differentiation (隱函數微分)

The Chain Rule can be used to give a more complete description of the process of implicit differentiation. Suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = y(x)$, where $F(x, y(x)) = 0$ for all x in the domain of y. If F is differentiable, we can apply the Chain Rule to differentiate both side of the equation $F(x, y(x)) = 0$ with respect to x to get

$$\Rightarrow$$

Implicit Function Theorem (page 942). *If $F(x, y)$ is defined on a disk containing (x_0, y_0), where $F(x_0, y_0) = 0, F_y(x_0, y_0) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (x_0, y_0) and the derivative of this function is $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$.*

Exercise (page 946). If $f(x, y) = 0$ define y as a function of x, show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3}.$$

Now we suppose that z is given implicitly as a function $z = z(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, z(x, y)) = 0$ for all (x, y) in the domain of z. If F and z are differentiable, then we can use the Chain Rule to differentiable the equation $F(x, y, z(x, y)) = 0$ as follows:

If $\frac{\partial F}{\partial z} \neq 0$, we solve $\frac{\partial z}{\partial x}$ and obtain

$$\frac{\partial z}{\partial x} = \qquad\qquad \frac{\partial z}{\partial y} =$$

Implicit Function Theorem (page 943). *If $F(x, y)$ is defined within a sphere containing (x_0, y_0, z_0), where $F(x_0, y_0) = 0$, $F_z(x_0, y_0, z_0) \neq 0$, and F_x, F_y and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (x_0, y_0, z_0) and the partial derivatives of this function are $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.*

Example 6 (page 943). Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 = 0$.

Solution.