14.4 Tangent Planes and Linear Approximations, page 927

Tangent Planes, page 928

Definition 1 (page 928)**.** Suppose that a surface S has equation $z = f(x, y)$, where f has *continuous partial derivatives*, and let $P(x_0, y_0, z_0)$ be a point on S. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at P. Then the *tangent plane* (切平面) to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.

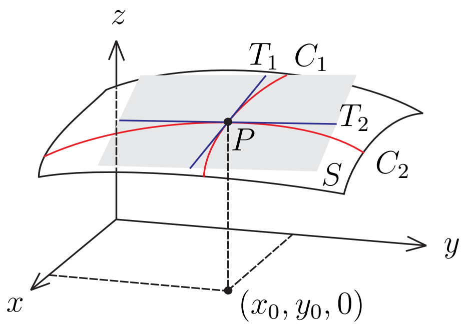

Figure 1: The tangent plane contains the tangent lines T_1 and T_2.

An equation of the tangent plane to the surface $z = f(x, y)$ at $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0), \quad \text{or} \qquad \text{(點斜式)}$$
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0 \qquad \text{(用法向量看待)}$$

Remark 2. Since tangent vectors to C_1 and C_2 at P are $\mathbf{e}_1 = 1\,\mathbf{i} + 0\,\mathbf{j} + f_x(x_0, y_0)\,\mathbf{k}$ and $\mathbf{e}_2 = 0\,\mathbf{i} + 1\,\mathbf{j} + f_y(x_0, y_0)\,\mathbf{k}$, a normal vector of the tangent plane is

$$\begin{aligned}\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2 &= -f_x(x_0, y_0)\,\mathbf{i} - f_y(x_0, y_0)\,\mathbf{j} + 1\,\mathbf{k} \\ &\mathbin{/\!/} f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} - 1\,\mathbf{k}.\end{aligned}$$

☐ 若函數具有「連續偏導數」(f_x 與 f_y 是連續函數), 才有切平面。

Example 3. Find the equation of the tangent plane of the surface $z = e^{x-y}$ at the point $P(1, 1, 1)$.

Solution.

Linear Approximations, page 929

Definition 4 (page 929)**.** An equation of the tangent plane to the graph of the function $z = f(x, y)$ at $P(x_0, y_0, z_0)$ is $z - z_0 = z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. The linear function whose graph is this tangent plane, namely,

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called *linearization* (線性化) of f at (x_0, y_0) and the approximation

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{1}$$

is called the *linear approximation* (線性估計) or *tangent plane approximation* of f at (x_0, y_0).

Example 5 (page 930)**.** Consider the function $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

(a) $f_x(0, 0) =$

(b) $f_y(0, 0) =$

(c) We take the path $C_1(t) = (t, t), t \neq 0$, the function $f(x, y)|_{C_1(t)} =$

(d) A function of two variables can behave badly even through both of its partial derivatives exist. To rule out such behavior, we will define a *differentiable function* (可微分函數) of two variable.

Definition 6 (page 931)**.** If $z = f(x, y)$, then f is *differentiable* (可微分的) at (x_0, y_0) if $\Delta x = x - x_0, \Delta y = y - y_0$, then $f(x, y)$ satisfies

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f(x, y) - f(x_0, y_0) - f_x(x_0, y_0)\Delta x - f_y(x_0, y_))\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0.$$

Sometimes it is hard to use the definition to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

Theorem 7 (page 932)**.** *If the partial derivatives f_x and f_y exist near (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).*

☐ 函數具有連續偏導數, 切平面相應的線性函數才是好的線性估計。

☐ 多變數函數, 導數 (derivative) 與可微分 (differentiable) 兩者概念上有別。

Differentials, page 932

For a differentiable function of two variables, $z = f(x, y)$, we define the *differentials* (微分) $\mathrm{d}x$ and $\mathrm{d}y$ to be independent variables; that is, they can be given any values. Then the *differential* $\mathrm{d}z$, also called the *total differential* (全微分), is defined by

$$\mathrm{d}z = \mathrm{d}f = f_x(x, y)\,\mathrm{d}x + f_y(x, y)\,\mathrm{d}y = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y. \qquad (2)$$

If we take $\mathrm{d}x = \Delta x = x - x_0$ and $\mathrm{d}y = \Delta y = y - y_0$ in (2), then the differential of z is $\mathrm{d}z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$, so in notation of differentials, the linear approximation (1) can be written as $f(x, y) \approx f(x_0, y_0) + \mathrm{d}z$.

Figure 2 shows the geometric interpretation of the differential $\mathrm{d}x$ and the increment Δz: $\mathrm{d}z$ represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (x_0, y_0) to $(x_0 + \Delta x, y_0 + \Delta y)$.

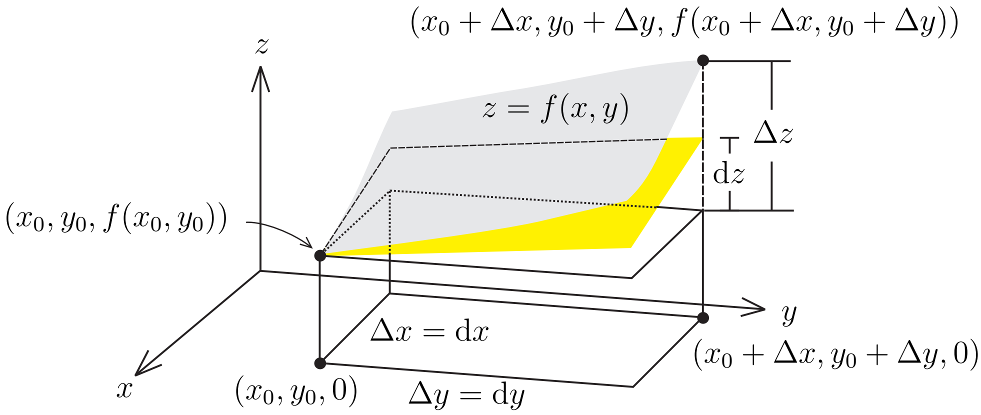

Figure 2: Geometric interpretation of the differential $\mathrm{d}z$ and the increment Δz.

Example 8 (page 933)**.** The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Solution.

Functions of Three or More Variables, page 932

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables.

Example 9. Let $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) $f(x,y)$ is continuous at $(0,0)$ because

(b) $f_x(0,0) =$

(c) $f_y(0,0) =$

(d) For $(x,y) \neq (0,0)$, $\frac{\partial f}{\partial x} =$

(e) $\frac{\partial f}{\partial x}(x,y)$ is *not* continuous at $(0,0)$ because we take the path $C_1(t) = (t,t), t \neq 0$, then the function $f_x(x,y)|_{C_1(t)} =$

(f) Compute for $(x,y) \neq (0,0)$

$$f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y =$$

and take the path $C_1(x) = (x,x), x \neq 0$, we find

$$f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y|_{C_1(x)} =$$

(g) Form (e) and (f), we know that $L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y \equiv 0$ is *not* a good linear approximation of $f(x,y)$ at $(0,0)$.