14.3 Partial Derivative, page 911

Definition 1 (page 913)**.** If f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = y_0$, then $g(x) = f(x, y_0)$ is a function of a single variable x. If $g(x)$ has a derivative at $x = x_0$, then we call it the *partial derivative* (偏導數) *of* f *with respect to* x *at* (x_0, y_0) and denote it by $f_x(x_0, y_0)$. Thus

$$f_x(x_0, y_0) = g'(x_0) = \lim_{h\to 0} \frac{g(x_0 + h) - g(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

Similarly, the *partial derivative* (偏導數) *of* f *with respect to* y *at* (x_0, y_0) and denote it by $f_y(x_0, y_0)$, is obtained by keeping x fixed, say $x = x_0$, and finding the ordinary derivative at $y = y_0$ of the function $\tilde{g}(y) = f(x_0, y)$:

$$f_y(x_0, y_0) = \tilde{g}'(y_0) = \lim_{h\to 0} \frac{\tilde{g}(y_0 + h) - \tilde{g}(y_0)}{h} = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Definition 2 (page 913)**.** If f is a function of two variables, its *partial derivatives* (偏導函數) are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h},$$
$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Notations for Partial Derivatives. If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_x f = D_1 f,$$
$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_y f = D_2 f.$$

Rule for Finding Partial Derivative of $z = f(x, y)$.

(1) To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

(2) To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

☐ 對某變數求偏導, 固定其他變數, 使其爲單變數函數, 再計算導數。

Example 3 (page 914)**.** If $f(x, y) = x^3 + x^2 y^3 - 2y^2$, then

(a) $f_x(x, y) =$

$f_x(2, 1) =$

(b) $f_y(x, y) =$

$f_y(2, 1) =$

Interpretations of Partial Derivatives, page 915

The partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ can be interpreted geometrically as the slopes of the tangent lines at $P(x_0, y_0, f(x_0, y_0))$ to the trace C_1 and C_2 of the surface S in the planes $y = y_0$ and $x = x_0$.

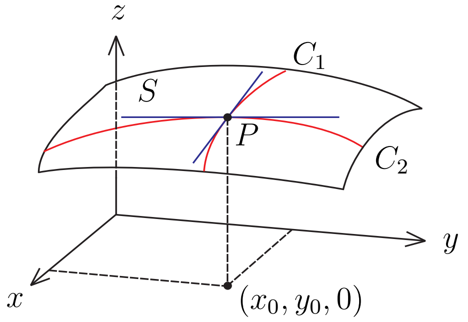

Figure 1: Geometric meaning of partial derivatives.

Example 4 (page 917)**.** If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution.

Example 5 (page 917)**.** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution.

Exercise (page 927)**.** If $f(x, y) = \dfrac{x\, e^{\sin(x^2 y)}}{(x^2 + y^2)^{\frac{3}{2}}}$, find $f_x(1, 0)$.

Functions of More Than Two Variables, page 917

If $z = f(x_1, x_2, \ldots, x_n)$ is a function of n variables, its partial derivative with respect to the i-th variable x_i is

$$\frac{\partial z}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}.$$

We also write

$$\frac{\partial z}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f.$$

Higher Derivatives, page 918

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x, (f_x)_y, (f_y)_x$, and $(f_y)_y$, which are called the *second partial derivatives* (二次偏導數) of f. If $z = f(x, y)$, we use the following notation:

$$
\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}, \\
(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}, \\
(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}, \\
(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}.
\end{aligned}
$$

☐ 寫成下標的順序 f_{xy} 和寫成 $\frac{\partial^2 f}{\partial y \partial x}$ 的順序及其代表之意義需注意。

Exercise. Let $r(x, y) = \sqrt{x^2 + y^2}$. For $(x, y) \neq (0, 0)$, compute $r_x, r_y, r_{xx}, r_{xy}, r_{yx}$, and r_{yy}.

Clairaut's Theorem (page 919)**.** *Suppose f is defined on a disk D that contains the point (x_0, y_0). If the functions f_{xy} and f_{yx} are both continuous on D, then*

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

☐ 二次偏導函數 f_{xy} 與 f_{yx} 必須都是「連續函數」, 偏導順序交換才會相等。

Exercise. Let $f(x, y) = \dfrac{x^3 - xy^2}{x^2 + y^2}$.

(a) Determine the value $f(0, 0)$ such that $f(x, y)$ is continuous at $(0, 0)$.

(b) Find $f_x(x, y), f_x(x, y), f_x(0, 0)$ and $f_y(0, 0)$.

(c) Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

Partial Differential Equations, page 920

Partial derivatives occur in *partial differential equations* (偏微分方程) that express certain physical laws. For instance,

(a) $u = u(x, y), \Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0$: Laplace's equation (拉普拉斯方程).

(b) $u = u(t, x), \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$: heat equation. (熱傳導方程).

(c) $u = u(t, x), \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$: wave equation (波動方程)

Example 6 (page 927)**.** Let $f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

(a) Find $f_x(x,y)$ and $f_y(x,y)$ when $(x,y) \neq (0,0)$.

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

(c) Find $f_{xy}(x,y)$ and $f_{yx}(x,y)$ when $(x,y) \neq (0,0)$.

(d) Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

(e) Do the results of (c) and (d) contradict Clairaut's Theorem?

Solution.

(a) Direct computation gives

$$f_x(x,y) = \frac{(x^2+y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$f_y(x,y) = \qquad .$$

(b) By definition, we have

$$f_x(0,0) =$$

$$f_y(0,0) =$$

(c) Direct computation gives

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

$$f_{yx}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}.$$

(d) By definition, we have

$$f_{xy}(0,0) =$$

$$f_{yx}(0,0) =$$

(e) Results of (c) and (d) don't contradict to Clairaut's Theorem because both $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are not continuous at $(0,0)$. We cant take path $C_1(x) = (x,0), x \neq 0$ and $C_2(y) = (0,y), y \neq 0$ to get $f_{xy}(x,y)|_{C_1(x)} = f_{yx}(x,y)|_{C_1(x)} \equiv 1$ and $f_{xy}(x,y)|_{C_2(y)} = f_{yx}(x,y)|_{C_2(y)} \equiv -1$. That is, $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ and $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ do <u>not</u> exist.

□ 分段函數求偏導, 用定義計算。由 (d) 知, 二次偏導函數順序交換不見得相等。