14.2 Limits and Continuity, page 903

Definition 1 (page 904)**.** Let f be a function of two variables whose domain D includes points arbitrary close to (a, b). Then we say that the *limit of* $f(x, y)$ *as* (x, y) *approaches* (a, b) is L (函數 $f(x, y)$ 靠近 (a, b) 的極限值是 L) and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \varepsilon$.

Other notations for the limit are

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y) = L \qquad \text{and} \qquad f(x, y) \to L \text{ as } (x, y) \to (a, b).$$

The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Therefore, we get

Property 2 (page 905)**.** *If* $f(x, y) \to L_1$ *as* $(x, y) \to (a, b)$ *along a path* C_1 *and* $f(x, y) \to L_2$ *as* $(x, y) \to (a, b)$ *along a path* C_2*, where* $L_1 \neq L_2$*, then* $\lim_{(x,y)\to(a,b)} f(x, y)$ *does not exist.*

☐ 極限存在 ⇔ 以以及「任何路徑」靠近都要接近一個明確的值。

Example 3 (page 905)**.** Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Solution.

Example 4 (page 906)**.** If $f(x, y) = \frac{xy}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

Solution.

Example 5 (page 906)**.** If $f(x,y) = \frac{xy^2}{x^2+y^4}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution.

□ 上例得知如果只有以「各種角度」直線靠近一個明確的值, 極限仍有可能不存在。

We can use polar coordinates to find the limit. Note that if (r,θ) are polar coordinates of the point (x,y) with $r \geq 0$, then $r \to 0^+$ as $(x,y) \to (0,0)$.

Example 6 (page 896)**.** Find $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Solution.

Exercise (page 910–911)**.** Find the limit, if it exists, or show that the limit does not exist.

(a) $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

(b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y\,\mathrm{e}^y}{x^4+4y^2}$.

(c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$.

(d) $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

Continuity, page 907

Definition 7 (page 908)**.** A function f of two variables is called *continuous at* (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b) = f\left(\lim_{(x,y)\to(a,b)} x, \lim_{(x,y)\to(a,b)} y\right).$$

We say f is *continuous on* D if f is continuous at every point (a,b) in D.

A *polynomial function of two variables* (or *polynomial* 二變數多項式, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A *rational function* (有理函數) is a ratio of polynomials. All polynomials are continuous on $\mathbb{R}^2$. Any rational function is continuous on its domain because it is a quotient of continuous functions.

Example 8 (page 908)**.** Where is the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ continuous?

Solution. The function $f(x,y)$ is discontinuous at $(0,0)$ because it is not defined there. Since $f(x,y)$ is a rational function, it is continuous on its domain $D = \{(x,y)|(x,y) \neq (0,0)\}$.

Example 9 (page 908)**.** Let

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

Here $f(x,y)$ is defined at $(0,0)$ but $f(x,y)$ is still discontinuous there because $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. (See **Example 3**.)

Example 10. Let

$$f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

We know $f(x,y)$ is continuous for $(x,y) \neq (0,0)$ since it is equal to a rational function there. From **Example 6**, we have

Therefore, ________________________, and so it is continuous on ___.

Property 11 (page 909)**.** *If f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x,y) = g(f(x,y))$ is also a continuous function.*

Example 12 (page 909)**.** Where is the function $h(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ continuous?

Solution. The function $f(x,y) = \frac{y}{x}$ is a rational function and therefore continuous except on ______. The function $g(t) = \tan^{-1} t$ is continuous everywhere, so the composition function $h(x,y) = g(f(x,y)) = \tan^{-1}\left(\frac{y}{x}\right)$ is continuous except where ______.

Exercise (page 911)**.** Determine the set of points at which the function is continuous.

$$f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases} .$$

Functions of Three or More Variables, page 909

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$$

means that the values of $f(x,y,z)$ approach the number L as the point (x,y,z) approaches the point (a,b,c) along *any* path in the domain of f. The function f is *continuous* at (a,b,c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c) = f\left(\lim_{(x,y,z)\to(a,b,c)} x, \lim_{(x,y,z)\to(a,b,c)} y, \lim_{(x,y,z)\to(a,b,c)} z\right).$$

For a function of n variables, we can write these definitions in a single compact form by vector notation. For instance, let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, and $f(\mathbf{x})$ is a function of n variable, The function f is *continuous* at $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}) = f\left(\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{x}\right).$$

Exercise. Let

$$f(x,y,z) = \begin{cases} \frac{xy+yz^3}{x^2+z^6} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0). \end{cases}$$

Determine the set of points at which $f(x,y,z)$ is continuous.