Chapter 14 Partial Derivatives

14.1 Functions of Several Variables, page 888

Functions of Two Variable, page 888

Definition 1 (page 888)**.** A *function f of two variables* (雙變數函數) is a rule that assigns to each ordered pair of real numbers (x, y) in a set $D \subset \mathbb{R}^2$ a unique real number denoted by $f(x, y)$. The set D is the *domain* (定義域) of f and its *range* (值域) is the set of values that f takes on, that is, $\{f(x, y) | (x, y) \in D\}$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are *independent variables* (獨立變數) and z is the *dependent variable* (依賴變數).

Example 2 (page 888–889)**.**

(a) Function: $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$. Domain: $D = \{(x, y) | x + y + 1 \geq 0, x \neq 1\}$.

(b) Function: $g(x, y) = x \ln(y^2 - x)$. Domain: $D = \{(x, y) | x < y^2\}$.

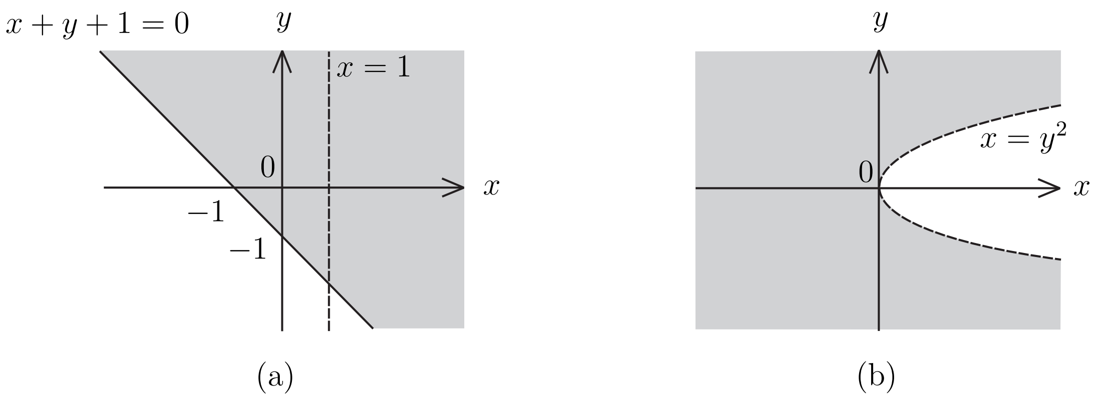

Figure 1: (a) Domain of $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$. (b) Domain of $g(x, y) = x \ln(y^2 - x)$.

Exercise. Find and sketch the domain of the function $f(x, y) = \sin^{-1}(x^2 + y^2 - 2)$.

Graphs, page 890

One way of visualizing the behavior of a function of two variables is to consider its graph.

Definition 3 (page 890)**.** If f is a function of two variables with domain D, then the *graph* of f (圖形) is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

The graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane.

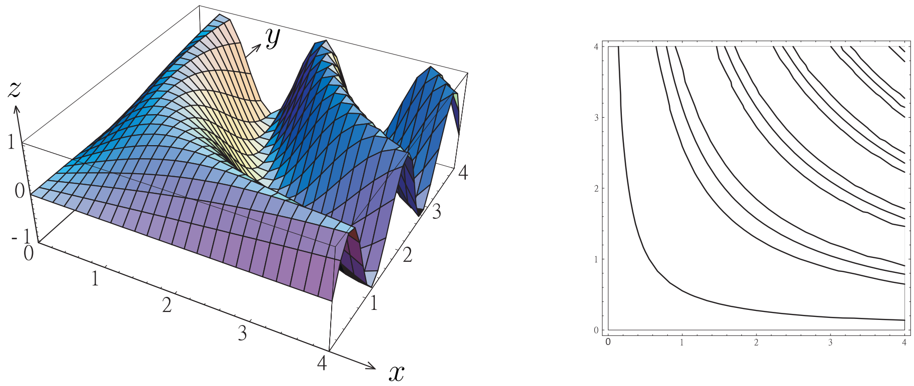

Figure 2: The graph of $f(x, y) = \sin xy, 0 \le x \le 4, 0 \le y \le 4$ and its level curves.

Level Curves, page 893

Another method for visualizing functions, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour lines* (等高線、輪廓線), or *level curves* (等位線).

Definition 4 (page 893)**.** The *level curves* of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane. The surface is steep where the level curves are close together. It is somewhat flatter where they are farther apart.

☐ 等高線、等壓線、等溫線。

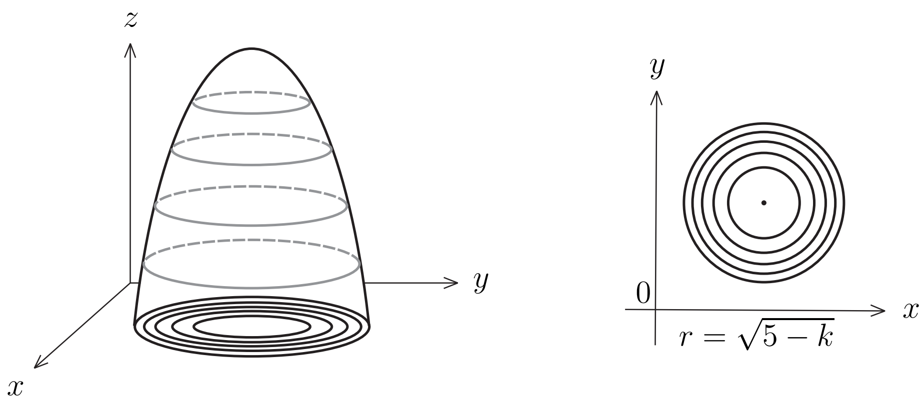

Figure 3: Level curves of the function $f(x, y) = 5 - (x - 3)^2 - (y - 3)^2$.

Exercise (page 902). Match the function (a),(b),(c) with its graph (A),(B),(C) and its contour map (Ⅰ), (Ⅱ), (Ⅲ). Give reasons for your choices.

(a) $f(x,y) = \sin x - \sin y$ (b) $g(x,y) = \dfrac{x-y}{1+x^2+y^2}$ (c) $h(x,y) = \mathrm{e}^x \cos y$.

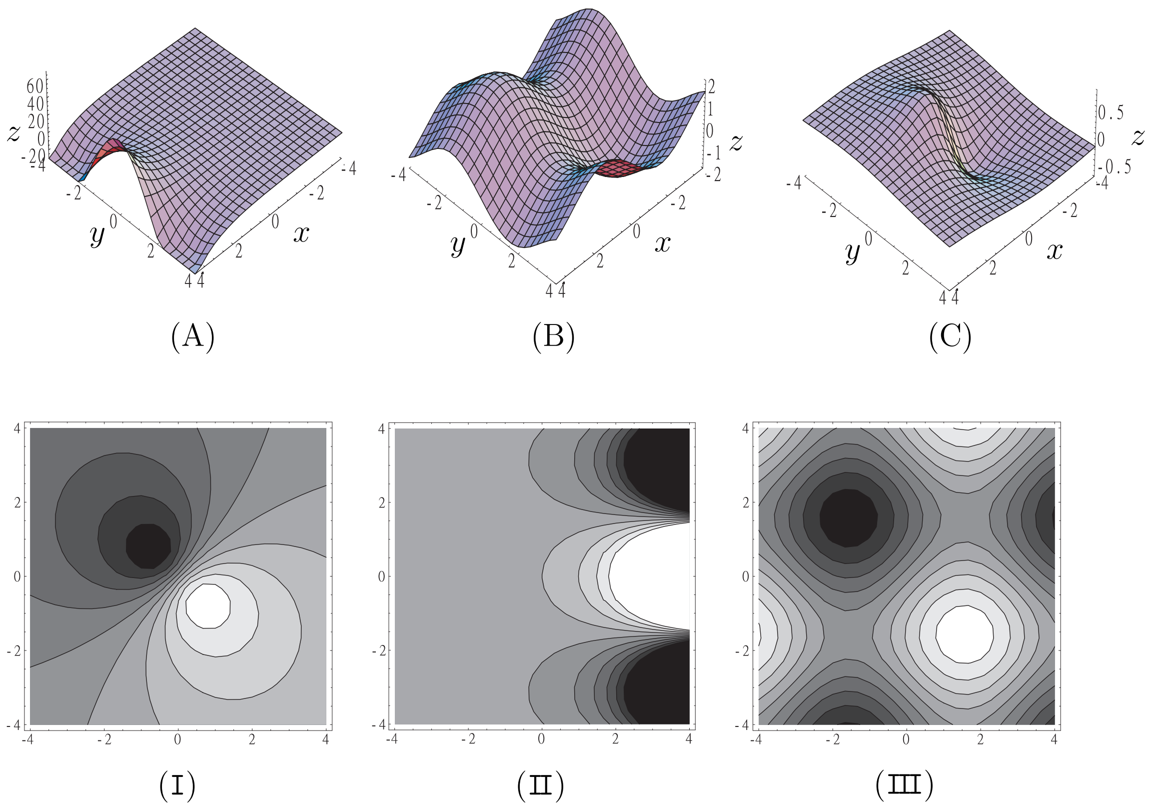

Figure 4: Match functions, graphs, and contour maps.

Functions of Three or More Variables, page 89

A *function of three variables* (三變數函數), f, is a rule that assigns to each ordered triple (x,y,z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x,y,z) \in \mathbb{R}$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time, so we could write $T = f(x,y,t)$.

In general, a *function of n variables* (n-變數函數) is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. Sometimes are will use vector notation to write such functions more compactly: If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$.