13.3 Arc Length and Curvature (page 861)

Question. How do we know that two space curves are the same (congruent)?

Concept of a Curve

Definition 1 (page 861)**.** Suppose that C is a space curve given by a vector function

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}, \quad a \le t \le b.$$

We say $\mathbf{r}(t)$ is a *smooth parametrization* (光滑的參數表示法) if

(a) $\mathbf{r}'(t)$ is continuous on $[a, b]$. $(\Leftrightarrow f(t), g(t), h(t) \in C^1[a, b])$

(b) $\mathbf{r}'(t) \neq \mathbf{0}$ for all $t \in [a, b]$. $(\Leftrightarrow \forall t \in [a, b]$, at least one of $f'(t), g'(t), h'(t) \neq 0)$

The variable t is called the *parameter* (參數) of the representation.

☐ 我們要利用「向量函數」或「參數式」(本質是函數) 研究「空間曲線」(本質是集合)。

Definition 2 (page 863)**.** A space curve C is called *smooth curve* (光滑曲線) if it has a smooth parametrization.

A smooth curve has *no* sharp corners or cusps; when the tangent vector turns, it does so continuously.

Definition 3 (page 862)**.** A space curve C can be smooth parametrized by more than one vector function. We say all vector functions are *parametrizations* (參數表示法) of the curve C.

Example 4 (page 862)**.** The twisted cubic $\mathbf{r}_1(t) = (t, t^2, t^3), 1 \le t \le 2$ could also be represented by the function $\mathbf{r}_2(u) = (\mathrm{e}^u, \mathrm{e}^{2u}, \mathrm{e}^{3u}), 0 \le u \le \ln 2$, where $t = \mathrm{e}^u$.

Example 5 (page 854)**.** A plane curve can be thought as a special case of a space curve. So we have many parametrizations to represent a unit circle $x^2+y^2 = 1, z = 0$. For example, $\mathbf{r}_1(t) = (\cos t, \sin t, 0), 0 \le t \le 2\pi$, or $\mathbf{r}_2(u) = (\cos 2u, \sin 2u, 0), 0 \le u \le \pi$, where $t = 2u$.

☐ 有些書籍或文獻是用正則曲線 (regular curve) 一詞取代光滑曲線 (smooth curve)。

☐ 同一曲線有很多參數法, 物理上稱爲 gauge invariance, 幾何上稱爲 diffeomorphism。

☐ 這些不同參數法, 之間的差異爲何? 有沒有「好的」或「自然的」表示方式?

Arc Length as a Parameter, page 863

Definition 6 (page 863)**.** Suppose that C is a simple smooth curve given by

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}, \quad a \le t \le b.$$

We define its *arc length function* s (弧長函數) by

$$s = s(t) = \int_a^t |\mathbf{r}'(u)|\,\mathrm{d}u = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}\,\mathrm{d}u. \tag{1}$$

Thus $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$.

By the Fundamental Theorem of Calculus, we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \tag{2}$$

We know $s(t)$ is an increasing function, and it is a one-to-one function, so its inverse function $t(s)$ exists and

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{\frac{\mathrm{d}s}{\mathrm{d}t}} \tag{3}$$

is a continuous function. Hence for a space curve C given by a vector function

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}, \quad a \le t \le b,$$

it can be reparametrized by arc length function

$$\mathbf{r}(s) = \mathbf{r}(t(s)) = f(t(s))\,\mathbf{i} + g(t(s))\,\mathbf{j} + h(t(s))\,\mathbf{k}, \quad c \le s \le d,$$

where $t(c) = a$ and $t(d) = b$. We will show that $\mathbf{r}(s)$ is a smooth parametrization:

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{d}f(t)}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\,\mathbf{i} + \frac{\mathrm{d}g(t)}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\,\mathbf{j} + \frac{\mathrm{d}h(t)}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\,\mathbf{k},$$

so $\mathbf{r}(s)$ is a nonzero, continuous vector function.

Thus the arc length s can be introduced along the curve as a parameter. It is often useful to *parametrize a curve with respect to arc length* (以弧長爲參數) because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system.

☐ 弧長參數是「好的」且「自然的」參數表示法。

Definition 7 (page 862)**.** The length of a space curve C is the limit of lengths of inscribed polygons. Suppose that a simple smooth curve has the vector equation $\mathbf{r}(t) = (f(t), g(t), h(t))$, $a \le t \le b$, where $f'(t), g'(t)$, and $h'(t)$ are continuous. Then its *length* (曲線長度) is

$$L = \int_a^b |\mathbf{r}'(t)|\,\mathrm{d}t = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,\mathrm{d}t.$$

A space curve C has many different parameterizations. We have to show that the length is well-defined: Suppose that $\mathbf{r}(u)$ is another smooth parametrization, and $u = u(t), t = t(u), c \le u \le d, t(c) = a$, and $t(d) = b$, then $t'(u) > 0$, and

$$L = \int_a^b \left| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right| \mathrm{d}t = \int_c^d \left| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right| \frac{\mathrm{d}t}{\mathrm{d}u} \,\mathrm{d}u = \int_c^d \left| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} \right| \mathrm{d}u.$$

☐ 空間曲線「長度」的定義是良好的 (well-defined); 不同的參數表示, 其結果都一樣。

Example 8 (page 863)**.** Reparametrize the helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t. Find the length of the arc of the circular helix from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

Solution.

Curvature, page 863

If C is a smooth curve defined by the vector function $\mathbf{r}(t)$, recall that the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and indicated the direction of the curve.

Remark that if a space curve is parametrized by arc length $\mathbf{r}(s) = \mathbf{r}(t(s))$, then

$$\left| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} \right| = \left| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right| \left| \frac{\mathrm{d}t}{\mathrm{d}s} \right| = \left| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right| \frac{1}{\left| \frac{\mathrm{d}s}{\mathrm{d}t} \right|} = \left| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right| \frac{1}{\left| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right|} = 1.$$

That is, if a space curve C is parametrized by arc length $\mathbf{r}(s)$, then $\mathbf{r}'(s)$ is unit tangent vector.

☐ 若曲線以「弧長」爲參數, 其切向量長度就是 1, 這也是弧長參數「好的」性質。

Definition 9 (page 864)**.** The *curvature* (曲率) of a space curve is

$$\kappa = \left| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

☐ 曲率是對「弧長參數」求導, 若用一般參數表示時, 必須要用鏈鎖率 (chain rule)。

☐ 因爲「弧長參數」是最自然的參數, 所以幾何量的定義都以「弧長參數」爲依據。

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length.

Theorem 10 (page 864–869)**.**

(a) *If t is another parameter instead of the arc length s, then*

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

(b) *The curvature of the curve given by the vector function $\mathbf{r}(t)$ is*

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

(c) *If a plane curve is given as the graph of a function $y = f(x)$, then the curvature of the curve is*

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{\frac{3}{2}}}.$$

(d) *If a plane curve is given as a plane parameter $x = x(t), y = y(t)$, then the curvature of the curve is*

$$\kappa = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{((x'(t))^2 + (y'(t))^2)^{\frac{3}{2}}}.$$

(e) *If a plane curve is given as $r = r(\theta), a \leq \theta \leq b$ in polar coordinates, then the curvature of the curve is*

$$\kappa(\theta) = \frac{|2(r'(\theta))^2 - r(\theta)r''(\theta) + r^2(\theta)|}{((r'(\theta))^2 + (r(\theta))^2)^{\frac{3}{2}}}.$$

Proof.

(a) By chain rule, we have

$$\kappa \overset{\text{def.}}{=} \left|\frac{\mathrm{d}\mathbf{T}(s)}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\mathbf{T}(t(s))}{\mathrm{d}s}\right| =$$

(b) Since $\mathbf{r}'(t) = \mathbf{T}(t)|\mathbf{r}'(t)|$, we have

$$\mathbf{r}''(t) =$$

and

$$\mathbf{r}'(t) \times \mathbf{r}''(t) =$$

Notice that $|\mathbf{T}(t)| = 1$, and it implies $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal, so

$$\frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} =$$

(c) A plane curve can be parametrized by $(x, f(x), 0)$ in $\mathbb{R}^3$, that is,

$$\mathbf{r}(x) = x\,\mathbf{i} + f(x)\,\mathbf{j} + 0\,\mathbf{k}.$$

We compute

$$\begin{aligned} \mathbf{r}'(x) &= 1\,\mathbf{i} + f'(x)\,\mathbf{j} + 0\,\mathbf{k} \qquad |\mathbf{r}'(x)| = \sqrt{1 + (f'(x))^2} \\ \mathbf{r}''(x) &= 0\,\mathbf{i} + f''(x)\,\mathbf{j} + 0\,\mathbf{k} \\ \mathbf{r}'(x) \times \mathbf{r}''(x) &= \end{aligned}$$

so the curvature is

$$\kappa(x) = \frac{|\mathbf{r}'(x) \times \mathbf{r}''(x)|}{|\mathbf{r}'(x)|^3} =$$

(d) A plane curve can be parametrized by $(x(t), y(t), 0)$ in $\mathbb{R}^3$, that is,

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + 0\,\mathbf{k}.$$

We compute

$$\begin{aligned} \mathbf{r}'(t) &= x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + 0\,\mathbf{k} \qquad |\mathbf{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2} \\ \mathbf{r}''(t) &= x''(t)\,\mathbf{i} + y''(t)\,\mathbf{j} + 0\,\mathbf{k} \\ \mathbf{r}'(t) \times \mathbf{r}''(t) &= \end{aligned}$$

so the curvature is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} =$$

(e) A plane curve can be parametrized by $(r(\theta)\cos\theta, r(\theta)\sin\theta, 0)$ in $\mathbb{R}^3$, that is,

$$\mathbf{r}(\theta) = r(\theta)\cos\theta\,\mathbf{i} + r(\theta)\sin\theta\,\mathbf{j} + 0\,\mathbf{k}.$$

We compute

$$\begin{aligned} \mathbf{r}'(\theta) &= (r'\cos\theta - r\sin\theta)\,\mathbf{i} + (r'\sin\theta + r\cos\theta)\,\mathbf{j} + 0\,\mathbf{k} \quad |\mathbf{r}'(\theta)| = \sqrt{r^2 + (r')^2} \\ \mathbf{r}''(\theta) &= (r''\cos\theta - 2r'\sin\theta - r\cos\theta)\,\mathbf{i} + (r''\sin\theta + 2r'\cos\theta - r\sin\theta)\,\mathbf{j} + 0\,\mathbf{k}, \end{aligned}$$

so $\mathbf{k}$-component of $\mathbf{r}'(\theta) \times \mathbf{r}''(\theta)$ is

$$\begin{aligned} &(r'\cos\theta - r\sin\theta)(r''\sin\theta + 2r'\cos\theta - r\sin\theta) \\ &- (r'\sin\theta + r\cos\theta)(r''\cos\theta - 2r'\sin\theta - r\cos\theta) \\ =& -r''(\theta)r + 2(r'(\theta))^2 + r^2(\theta). \end{aligned}$$

The curvature is

$$\kappa(\theta) = \frac{|2(r'(\theta))^2 - r(\theta)r''(\theta) + r^2(\theta)|}{((r'(\theta))^2 + (r(\theta))^2)^{\frac{3}{2}}}.$$

□

Example 11 (page 864)**.** Show that the curvature of a circle of radius r is $\frac{1}{r}$.

Solution.

Example 12. Find the curvature of the circular helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$.

Solution.

The Normal and Binormal Vectors, page 866

Definition 13 (page 866)**.**

(a) We define the *principal unit normal vector* $\mathbf{N}(t)$ (or simply *unit normal* 主單位法向量) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

(b) The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the *binormal vector* (次法向量). It is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector.

Example 14 (page 866)**.** Find the unit normal and binormal vectors for the circular helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$.

Solution.

Definition 15 (page 870)**.**

(a) The plane determined by the normal vector $\mathbf{N}$ and binormal vector $\mathbf{B}$ at a point P on a curve C is called the *normal plane* (法平面) of C at P.

(b) The plane determined by the tangent vector $\mathbf{T}$ and normal vector $\mathbf{N}$ is called the *osculating plane* (密切平面) of C at P.

(c) The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which $\mathbf{N}$ points), and has radius $\rho = \frac{1}{\kappa}$ (the reciprocal of the curvature) is called the *osculating circle* (密切圓) (or the circle of curvature) (曲率圓) of C at P. The radius of the osculating circle is called the *radius of curvature* (曲率半徑) of C at P.

Torsion and Theory of Curve (Appendix)

Let $\mathbf{r}$ be a smooth curve parametrized by arc length s such that $\mathbf{r}''(s) \neq 0$. The number $\tau(s)$ defined by $\mathbf{B}'(s) = \tau(s)\mathbf{N}(s)$ is called the torsion of $\mathbf{r}(s)$.

To each value of the parameter s, we have associated three orthogonal unit vector $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$. The trihedron is called *Frenet trihedron* at s. Since $\mathbf{N} = \mathbf{B} \times \mathbf{T}$, we have

$$\mathbf{N}'(s) = \mathbf{B}'(s) \times \mathbf{T}(s) + \mathbf{B}(s) \times \mathbf{T}'(s) = -\tau(s)\mathbf{B}(s) - \kappa(s)\mathbf{T}(s).$$

Hence we get the *Frenet formulas*:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

Physically, we can think of a space curve as being obtained from a straight line by bending (curvature) and twisting (torsion). The following theorem states that κ and τ describe completely the local behavior of the curve.

Fundamental Theorem of the Local Theory of Curves. *Given smooth functions $\kappa(s) > 0$ and $\tau(s)$, there exists a smooth parametrized curve $\mathbf{r}(s)$ such that s is the arc length, $\kappa(s) > 0$ is the curvature, and $\tau(s)$ is the torsion of $\mathbf{r}(s)$. Moreover, any other curve $\bar{\mathbf{r}}$, satisfying the same condition, differs from $\mathbf{r}$ by a rigid motion; that is, there exists an orthogonal linear map $\mathbf{T}$ of $\mathbb{R}^3$, with positive determinant, and a vector $\mathbf{c}$ such that $\bar{\mathbf{r}} = \mathbf{T} \circ \mathbf{r} + \mathbf{c}$.*