13.2 Derivatives and Integrals of Vector Functions (page 855)

Derivatives, page 855

Definition 1 (page 855)**.** The *derivative* (導函數) $\mathbf{r}'(t)$ of a vector function $\mathbf{r}(t)$ is

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

Definition 2 (page 856)**.**

(a) The vector $\mathbf{r}'(t_0)$ is called the *tangent vector* (切向量) to the curve C defined by $\mathbf{r}(t)$ at the point $P = \mathbf{r}(t_0)$, provided that $\mathbf{r}'(t_0)$ exists and $\mathbf{r}'(t_0) \neq \mathbf{0}$.

(b) The *tangent line* (切線) to the curve C at $P = \mathbf{r}(t_0)$ is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t_0)$.

(c) The *unit tangent vector* (單位切向量) is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Theorem 3 (page 856)**.** *If* $\mathbf{r}(t) = (f(t), g(t), h(t)) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, *where* f, g, *and* h *are differentiable functions, then*

$$\mathbf{r}'(t) = (f'(t), g'(t), h'(t)) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}.$$

Theorem 4 (page 858)**.** *Suppose* $\mathbf{u}(t)$ *and* $\mathbf{v}(t)$ *are differentiable vector functions,* c *is a scalar, and* $f(t)$ *is a real-valued function. Then*

(1) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$.

(2) $\frac{\mathrm{d}}{\mathrm{d}t}(c\,\mathbf{u}(t)) = c\,\mathbf{u}'(t)$.

(3) $\frac{\mathrm{d}}{\mathrm{d}t}(f(t)\,\mathbf{u}(t)) = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$.

(4) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$.

(5) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

(6) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$.

Proof. Let $\mathbf{u}(t) = u_1(t)\,\mathbf{i} + u_2(t)\,\mathbf{j} + u_3(t)\,\mathbf{k}$ and $\mathbf{v}(t) = v_1(t)\,\mathbf{i} + v_2(t)\,\mathbf{j} + v_3(t)\,\mathbf{k}$.

(1)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t)+\mathbf{v}(t)) = \lim_{h\to 0}\frac{\mathbf{u}(t+h)+\mathbf{v}(t+h)-(\mathbf{u}(t)+\mathbf{v}(t))}{h}$$
$$= \lim_{h\to 0}\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h} + \lim_{h\to 0}\frac{\mathbf{v}(t+h)-\mathbf{v}(t)}{h} = \mathbf{u}'(t)+\mathbf{v}'(t).$$

(2)

$$\frac{\mathrm{d}}{\mathrm{d}t}(c\,\mathbf{u}(t)) = \lim_{h\to 0}\frac{c\,\mathbf{u}(t+h)-c\,\mathbf{u}(t)}{h} = c\lim_{h\to 0}\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h} = c\,\mathbf{u}'(t).$$

(4)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t)\cdot\mathbf{v}(t))$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}(u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t))$$
$$= u_1'(t)v_1(t)+u_2'(t)v_2(t)+u_3'(t)v_3(t)+u_1(t)v_1'(t)+u_2(t)v_2'(t)+u_3(t)v_3'(t)$$
$$= \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t).$$

(6)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(f(t))) = \frac{\mathrm{d}}{\mathrm{d}t}(u_1(f(t))\,\mathbf{i}+u_2(f(t))\,\mathbf{j}+u_3(f(t))\,\mathbf{k})$$
$$= u_1'(f(t))f'(t)\,\mathbf{i}+u_2'(f(t))f'(t)\,\mathbf{j}+u_3'(f(t))f'(t)\,\mathbf{k}$$
$$= (u_1'(f(t))\,\mathbf{i}+u_2'(f(t))\,\mathbf{j}+u_3'(f(t))\,\mathbf{k})f'(t) = \mathbf{u}'(f(t))f'(t).$$

□

Exercise. Show that

(3) $\frac{\mathrm{d}}{\mathrm{d}t}(f(t)\,\mathbf{u}(t)) = f'(t)\,\mathbf{u}(t)+f(t)\,\mathbf{u}'(t)$.

(5) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t)\times\mathbf{v}(t)) = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$.

Example 5 (page 858)**.** If $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}(t)\cdot\mathbf{r}(t) = c^2$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}(t)\cdot\mathbf{r}(t)) =$$

Thus $\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$, which says that ____________________________.

Exercise (page 861)**.** If $\mathbf{r}(t)\neq 0$, show that $\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}(t)| = \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{|\mathbf{r}(t)|}$.

Integrals, page 859

The *definite integral* (定積分) of a continuous vector function $\mathbf{r}(t)$ is

$$\begin{aligned}\int_a^b \mathbf{r}(t)\,\mathrm{d}t &= \lim_{n\to\infty}\sum_{i=1}^{n}\mathbf{r}(t_i^*)\Delta t\\ &= \lim_{n\to\infty}\left(\left(\sum_{i=1}^{n} f(t_i^*)\Delta t\right)\mathbf{i}+\left(\sum_{i=1}^{n} g(t_i^*)\Delta t\right)\mathbf{j}+\left(\sum_{i=1}^{n} h(t_i^*)\Delta t\right)\mathbf{k}\right)\\ &= \left(\int_a^b f(t)\,\mathrm{d}t\right)\mathbf{i}+\left(\int_a^b g(t)\,\mathrm{d}t\right)\mathbf{j}+\left(\int_a^b h(t)\,\mathrm{d}t\right)\mathbf{k}\end{aligned}$$

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \Big[\mathbf{R}(t)\Big]\Big|_{t=a}^{t=b} = \mathbf{R}(b)-\mathbf{R}(a),$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is, $\mathbf{R}'(t)=\mathbf{r}(t)$. We use the notation $\int \mathbf{r}(t)\,\mathrm{d}t$ for *indefinite integrals* (不定積分).