Chapter 13 Vector Functions

13.1 Vector Functions and Space Curves (page 848)

We now study functions whose values are vectors because such functions are needed to describe curves and surfaces in space.

Definition 1 (page 848)**.** A *vector-valued function*, or *vector function* (向量函數), is a function whose domain is a set of real numbers and whose range is a set of vectors.

Here we will focus on vector functions $\mathbf{r}$ whose values are three-dimensional vectors. This means that for every number t in the domain of $\mathbf{r}$ there is a unique vector in $\mathbb{R}^3$ denoted by $\mathbf{r}(t)$. We can write

$$\mathbf{r}(t) = (f(t), g(t), h(t)) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k},$$

where f, g, h are real-valued functions of t called the *component functions* (分量函數) of $\mathbf{r}$.

☐ 定義中 $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ 爲 $\mathbb{R}^3$ 中的直角坐標向量。

☐ 課本用尖括號 $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ 表示向量函數, 但大部分文獻仍使用小括號。

Example 2 (page 848)**.** If $\mathbf{r}(t) = (t^3, \ln(3-t), \sqrt{t})$, then the component functions are

$$f(t) = t^3, \qquad g(t) = \ln(3-t), \quad \text{and} \quad h(t) = \sqrt{t}.$$

By the usual convention, the *domain* (定義域) of $\mathbf{r}$ consists of all values of t for which the expresstion for $\mathbf{r}(t)$ is defined. Therefore the domain of $\mathbf{r}$ is $[0, 3)$.

Definition 3 (page 848)**.** The *limit* of a vector function $\mathbf{r}$ is defined by taking the limits of its components functions as follows. If $\mathbf{r}(t) = (f(t), g(t), h(t))$, then

$$\lim_{t\to a} \mathbf{r}(t) = \Big(\lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\Big) = \lim_{t\to a} f(t)\,\mathbf{i} + \lim_{t\to a} g(t)\,\mathbf{j} + \lim_{t\to a} h(t)\,\mathbf{k}$$

provided the limits of the component functions exist.

Definition 4 (page 849)**.** A vector function $\mathbf{r}$ is *continuous at a* if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

☐「連續」可以看成「函數」與「極限」可以互換。

☐ 向量函數 $\mathbf{r}$ (在 $t = a$) 連續若且唯若所有分量函數 $f(t), g(t), h(t)$ (在 $t = a$) 連續。

There is a closed connection between vector-valued functions and space curves.

Definition 5 (page 849)**.** The <u>set</u> C of all points (x, y, z) in space, where

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \tag{1}$$

and t varies throughout the interval I, is called a *space curve* (空間曲線). The equations in (1) are called *parametric equations of* C (參數方程) and t is called a *parameter* (參數).

☐ 有時候 $\mathbf{r}(t)$ 也稱爲位置向量 (position vector)。

☐ 「空間曲線」是 $\mathbb{R}^3$ 中的一些點所成的「集合」, 可能有很多不同的「參數方程」表達。

Example 6 (page 849)**.** The curve $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ is called a *helix* (螺旋線).

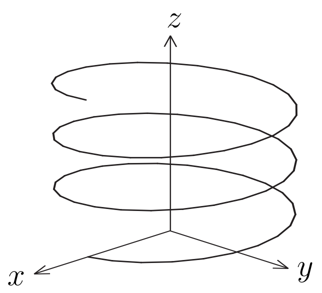

Figure 1: A helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}, 0 \le t \le 6\pi$.

Example 7.

(a) Find a vector equation and parametric equations for the line that join the point $A(a_1, a_2, a_3)$ to the point $B(b_1, b_2, b_3)$.

(b) Find a vector equation and parametric equations that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Solution.

Using computers to draw space curves, page 851

(a) Toroidal spiral:

$$x = (4 + \sin 7t)\cos t,\ y = (4 + \sin 7t)\sin t,\ z = \cos 7t,\ 0 \le t \le 2\pi.$$

(b) Trefoil knot:

$$x = (2 + \cos 1.5t)\cos t,\ y = (2 + \cos 1.5t)\sin t,\ z = \sin 1.5t,\ 0 \le t \le 4\pi.$$

(c) Twisted cubic: $x = t, \quad y = t^2, \quad z = t^3.$

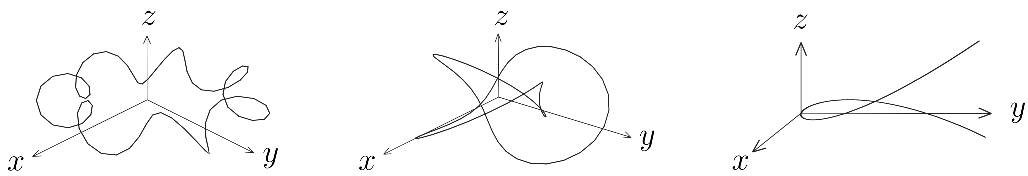

Figure 2: (a) Toroidal spiral. (b) Trefoil knot. (c) Twisted cubic.

Exercise (page 855)**.** Find a vector function that represents the curve of intersection of the two surfaces.

(a) The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$.

(b) The semiellipsoid $x^2 + y^2 + 4z^2 = 4, y \ge 0$, and the cylinder $x^2 + z^2 = 1$.