11.11 Applications of Taylor Polynomials (page 774)

In this section we explore some applications of Taylor polynomials. Computer scientists like them because polynomials are the simplest of functions. Physicists and engineers use them in such fields as relativity, optics, blackbody radiation, electric dipoles, the velocity of water waves, and building highways across a desert.

Approximating Functions by Polynomials, page 774

Recall that the *linear approximation* of $f(x)$ at $x = a$ (in section 3.10):

$$f(x) \approx f(a) + f'(a)(x - a) \tag{1}$$

Right hand side of (1), called the *linearization* of $f(x)$ at $x = a$, is the first-degree Taylor polynomial $T_1(x)$. If $f(x)$ is the sum of its Taylor series, then $T_n(x) \to f(x)$ as $n \to \infty$, and so $T_n(x)$, nth-degree Taylor polynomial of $f(x)$ at $x = a$, can be used as an approximation to $f(x)$:

$$f(x) \approx T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k.$$

When using a Taylor polynomial $T_n(x)$ to approximate a function $f(x)$, we have to ask that how good an approximation is it? How large should we take n to be in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder $|r_n(x)| = |R_n(x)| = |f(x) - T_n(x)|$. Here we remark that if $f(x)$ is the sum of its Taylor series, then $r_n(x) = R_n(x)$.

There are three possible methods for estimating the size of the error:

(1) If the series is an alternating series, we can use the Alternating Series Estimation Theorem.

(2) In all cases we can use Taylor Inequality: If $|f^{(n+1)}(x)| \le M$ for $|x - a| \le d$, then

$$|r_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1} \right| \le \frac{M}{(n+1)!}|x - a|^{n+1} \quad \text{for} \quad |x - a| \le d.$$

(3) If a graphing device is available, we can use it to graph (estimate) $|R_n(x)|$.

Example 1. Desmos Graphing Calculator is a free, online, graphing calculator:

`https://www.desmos.com/calculator`

`https://desmos.s3.amazonaws.com/Desmos_User_Guide.pdf`

We will illustrate Taylor polynomial approximations by Desmos.

Example 2 (page 775).

(a) Approximate $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 \leq x \leq 9$?

Solution.

(a) We compute

$$f(x) = \qquad f'(x) = \qquad f''(x) = \qquad f'''(x) =$$
$$f(8) = \qquad f'(8) = \qquad f''(8) =$$

So the desired approximation is

$$\sqrt[3]{x} \approx T_2(x) =$$
$$=$$

(b) We can use Taylor's Inequality with $n = 2$ at $x = 8$:

$$|r_2(x)| \leq$$
$$\leq$$

Thus, if $7 \leq x \leq 9$, the approximation in (a) is accurate to within __________.

Exercise. Approximate $\sqrt[5]{240}$ with error less than 0.0001.

Example 3 (page 776). What is the maximum error possible in using the approximation $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ when $-0.3 \leq x \leq 0.3$? Use this approximation to find $\sin 12°$ correct to six decimal places.

Solution. Notice that the Maclaurin series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ is alternating for all $x \neq 0$, and the successive terms decrease in size because $|x| < 1$, so we can use the __________________________________. The error in approximating $\sin x$ by the first three terms of its Maclaurin series is at most

$$\left|\frac{x^7}{7!}\right| = \frac{|x|^7}{5040} \leq$$

To find $\sin 12°$, we first *convert to radian measure*:

$$\sin 12° = \sin\left(12 \cdot \frac{\pi}{180}\right) = \sin\left(\frac{\pi}{15}\right)$$
$$\approx$$

Thus, correct to six decimal places, $\sin 12° \approx$ __________.

Applications to Physics, page 778

Example 4 (page 778)**.** In Einstein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest: $K = mc^2 - m_0 c^2$.

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2} m_0 v^2$.

(b) Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \leq 100\,\text{m/s}$.

Solution.

(a) Using the expressions given for K and m, we get

$$K = mc^2 - m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 = m_0 c^2 \left(\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1 \right).$$

With $x = -\frac{v^2}{c^2}$, the Maclaurin series for $(1+x)^{-\frac{1}{2}}$ is a binomial series with $m = -\frac{1}{2}$. Therefore we have

$$\begin{aligned}(1+x)^{-\frac{1}{2}} &= 1 - \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \cdots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots,\end{aligned}$$

and

$$\begin{aligned}K &= m_0 c^2 \left(\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots \right) - 1 \right) \\ &= m_0 c^2 \left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots \right)\end{aligned}$$

If v is much smaller than c, then all terms after the first are very small when compared with the first term. If we omit them, we get

$$K = m_0 c^2 \left(\frac{1}{2}\frac{v^2}{c^2} \right) = \frac{1}{2} m_0 v^2.$$

(b) Let $f(x) = m_0c^2\left((1+x)^{-\frac{1}{2}} - 1\right)$ with $x = -\frac{v^2}{c^2}$. We can use Taylor's Inequality to write

$$r_1(x) = \frac{f''(\tilde{c})}{2!}x^2, \quad \text{where } -\frac{v^2}{c^2} \le \tilde{c} \le 0.$$

Since $f''(x) = \frac{3}{4}m_0c^2(1+x)^{-\frac{5}{2}}$ and we are given that $|v| \le 100\,\text{m/s}$, so

$$|f''(\tilde{c})| = \frac{3m_0c^2}{4\left(1+\tilde{c}\right)^{\frac{5}{2}}} \le \frac{3m_0c^2}{4\left(1-\frac{100^2}{c^2}\right)^{\frac{5}{2}}}.$$

Thus, with $c = 3 \cdot 10^8\,\text{m/s}$,

$$|r_1(x)| = \frac{1}{2} \cdot \frac{3m_0c^2}{4\left(1-\frac{100^2}{c^2}\right)^{\frac{5}{2}}} \cdot \frac{100^4}{c^4} < (4.17 \cdot 10^{-10})m_0.$$

So when $|v| \le 100\,\text{m/s}$, the magnitude of the error in using the Newtonian expression for kinetic energy is at most $(4.17 \cdot 10^{-10})m_0$.

Exercise (page 782)**.** If a surveyor measures differences in elevation when making plans for a highway across a desert, corrections must be made for the curvature of the earth.

(a) If R is the radius of the earth and L is the length of the highway, show that the correction is

$$C = R\sec\left(\frac{L}{R}\right) - R.$$

(b) Use a Taylor polynomial to show that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3}.$$

(c) Compare the corrections given by the formulas in parts (a) and (b) for a highway that is 100 km long. (Take the radius of the earth to be 6370 km.)

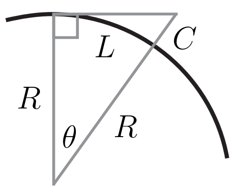

Figure 1: Surveyors measures differences in elevation of highway.

Exercise (page 785)**.** Find the sum of the series (a) $\sum\limits_{n=0}^{\infty} \frac{(-1)^n\pi^n}{3^{2n}(2n)!}$ (b) $\sum\limits_{n=0}^{\infty} \frac{(\ln 2)^{2n}}{(2n)!}$.

Appendix

Example 5. Consider the function

$$f(x) = \begin{cases} \mathrm{e}^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

(a) The function $f(x)$ is continuous on $\mathbb{R}$ because

$$\lim_{x\to 0} \mathrm{e}^{-\frac{1}{x^2}} = \lim_{y\to\pm\infty} \mathrm{e}^{-y^2} = \lim_{y\to\pm\infty} \frac{1}{\mathrm{e}^{y^2}} = 0 = f(0),$$

and for $x \neq 0$, $f(x)$ is a composition of two continuous functions $g(x) = \mathrm{e}^x$ and $h(x) = -\frac{1}{x^2}$, that is, $f(x) = (g \circ h)(x)$.

(b) We will show that: <u>For $x \neq 0$, $f^{(n)}(x) = P_n(y)\mathrm{e}^{-y^2}$</u>, where $y = \frac{1}{x}$, and $P_n(y)$ is a polynomial of y with degree $3n$.

(1) When $n = 1$, we compute

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-y^2}(-2y)\cdot(-y^2) = 2y^3\mathrm{e}^{-y^2} = P_1(y)\mathrm{e}^{-y^2},$$

where $P_1(y) = 2y^3$ is a polynomial of y with degree 3.

(2) Assume that it is true for $n = k$, that is, $f^{(k)}(x) = \frac{\mathrm{d}^k f}{\mathrm{d}x^k} = P_k(y)\mathrm{e}^{-y^2}$, where $P_k(y)$ is a polynomial with degree $3k$.

(3) When $n = k+1$, we compute

$$\begin{aligned} f^{(k+1)}(x) = \frac{\mathrm{d}^{k+1}f}{\mathrm{d}x^{k+1}} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}^k f}{\mathrm{d}x^k} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}^k f}{\mathrm{d}x^k}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}y}\left(P_k(y)\mathrm{e}^{-y^2}\right)(-y^2) \\ &= \left(\frac{\mathrm{d}P_k(y)}{\mathrm{d}y}\mathrm{e}^{-y^2} + P_k(y)\mathrm{e}^{-y^2}(-2y)\right)(-y^2) \\ &= \left(-y^2\frac{\mathrm{d}P_k(y)}{\mathrm{d}y} + 2y^3P_k(y)\right)\mathrm{e}^{-y^2}. \end{aligned}$$

Let $P_{k+1}(y) = -y^2\frac{\mathrm{d}P_k(y)}{\mathrm{d}y} + 2y^3P_k(y)$, which is a polynomial of y with degree $3 + 3k = 3(k+1)$.

(4) By mathematical induction, we know that for $x \neq 0$, $f^{(n)}(x) = P_n(y)\mathrm{e}^{-y^2}$, where $y = \frac{1}{x}$, and $P_n(y)$ is a polynomial of y with degree $3n$.

(c) Now, we will show that <u>$f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$</u>.

(1) When $n = 1$, we compute

$$\begin{aligned} f'(0) &= \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{\mathrm{e}^{-\frac{1}{x^2}}}{x} = \lim_{y\to\pm\infty}\frac{\mathrm{e}^{-y^2}}{\frac{1}{y}} \\ &= \lim_{y\to\pm\infty}\frac{y}{\mathrm{e}^{y^2}} \overset{(\frac{\infty}{\infty}),L'}{=} \lim_{y\to\pm\infty}\frac{1}{2y\mathrm{e}^{y^2}} = 0. \end{aligned}$$

(2) Assume that it is true for $n = k$, that is, $f^{(k)}(0) = 0$.

(3) When $n = k + 1$, we compute

$$f^{(k+1)}(0) = \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x\to 0} \frac{f^{(k)}(x)}{x} = \lim_{y\to\pm\infty} \frac{P_k(y)\mathrm{e}^{-y^2}}{\frac{1}{y}}$$
$$= \lim_{y\to\pm\infty} \frac{yP_k(y)}{\mathrm{e}^{y^2}} = 0.$$

Remark that we can apply L' Hospital Rule $\left[\frac{3n-1}{2}\right]$ times to get the limit is 0.

(4) By mathematical induction, we know that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

(d) Since $f(0) = 0$ and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, the Maclaurin series of $f(x)$ is

$$M(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = 0.$$

This is a zero function, so the interval of convergence of $M(x)$ is $\mathbb{R}$. We compute the remainder

$$r_n(x) = f(x) - T_n(x) = f(x).$$

We get for any $x \neq 0$, $\lim\limits_{n\to\infty} r_n(x) = \mathrm{e}^{-\frac{1}{x^2}} \neq 0$. Therefore, $f(x)$ is not equal to its Maclaurin series.

(e) For any integer $k \geq 0$, let $C^k(\mathbb{R})$ be the set (in fact, it is a vector space) consisting of all functions $f(x)$ that the derivatives $f'(x), f''(x), \ldots, f^{(k)}(x)$ exist and are continuous on $\mathbb{R}$. So $C^0(\mathbb{R})$, which is also denoted by $C(\mathbb{R})$, consists of all continuous functions on $\mathbb{R}$, and $C^\infty(\mathbb{R}) = \cap_{k=0}^{\infty} C^k(\Omega)$ consists of all smooth functions (continuous derivatives of all orders) on $\mathbb{R}$ (光滑函數).

Denote $C^\omega(\mathbb{R})$ be the set consisting of all smooth functions $f(x)$ that for all $x \in \mathbb{R}$, there exists $R > 0$ such that $f(x)$ equals its Taylor series expansion on $(x - R, x + R)$. We say a function $f(x) \in C^\omega(\mathbb{R})$ is *analytic* (解析函數).

(f) The above discussion shows that the function $f(x)$ is a smooth function, but not an analytic function because $f(x)$ is not analytic at $x = 0$. So the conclusion is $C^\omega(\mathbb{R}) \subsetneq C^\infty(\mathbb{R})$.

Remark that we have the following relations:

$$C^\omega(\mathbb{R}) \subsetneq C^\infty(\mathbb{R}) \cdots \subsetneq C^2(\mathbb{R}) \subsetneq C^1(\mathbb{R}) \subsetneq C^0(\mathbb{R}).$$

Example 6. Recall that the binomial series is

$$\sum_{n=0}^{\infty} C_n^m x^n = \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n.$$

We will check the convergence of the binomial series at the endpoints.

(a) <u>If $m \leq -1$</u>, then

$$\begin{aligned}|C_n^m x^n| = |C_n^m (\pm 1)^n| = |C_n^m| &= \left|\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}\right| \\ &= \frac{|m||(m-1)||(m-2)|\cdots|(m-n+1)|}{n!} \geq \frac{1\cdot 2\cdot 3\cdots n}{n!} = 1.\end{aligned}$$

So the series $\sum_{n=0}^{\infty} C_n^m x^n$ is divergent at $x = \pm 1$ by the Test oc Divergence.

(b_{-1}) <u>If $-1 < m < 0$ and $x = -1$</u>, then $0 < -m < 1$, and

$$\begin{aligned}C_n^m x^n &= \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}(-1)^n \\ &= \frac{(-m)(1-m)(2-m)\cdots(n-1-m)}{n!} \\ &= \frac{(-m)}{n}\cdot\frac{(1-m)}{1}\cdot\frac{(2-m)}{2}\cdots\frac{(n-1-m)}{n-1} \geq \frac{(-m)}{n}.\end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{(-m)}{n}$ is divergent (p-series, $p = 1$), $\sum_{n=0}^{\infty} C_n^m x^n$ is divergent at $x = -1$ by the Comparison Test.

(b_1) <u>If $-1 < m < 0$ and $x = 1$</u>, then $\sum_{n=0}^{\infty} C_n^m x^n = \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$ is an alternating series. We compute

$$\begin{aligned}|C_n^m| &= \left|\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}\right| \\ &\geq \left|\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}\right|\left|\frac{m-n}{n+1}\right| = |C_{n+1}^m|,\end{aligned}$$

so it is a decreasing sequence. Next, we calculate

$$\begin{aligned}|C_n^m| &= \left|\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}\right| \\ &= \left|\frac{m}{1}\cdot\frac{(m-1)}{2}\cdot\frac{(m-2)}{3}\cdots\frac{(m-n+1)}{n}\right| \\ &= \left|\left(1-\frac{m+1}{1}\right)\left(1-\frac{m+1}{2}\right)\cdots\left(1-\frac{m+1}{n}\right)\right| = \prod_{k=1}^{n}\left(1-\frac{m+1}{k}\right).\end{aligned}$$

Since

$$\ln|C_n^m| = \ln\left(\prod_{k=1}^{n}\left(1-\frac{m+1}{k}\right)\right) = \sum_{k=1}^{n}\ln\left(1-\frac{m+1}{k}\right) < \sum_{k=1}^{n} -\frac{m+1}{k}$$
$$= -(m+1)\sum_{k=1}^{n}\frac{1}{k}$$

and $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k} = \sum_{n=1}^{\infty}\frac{1}{n} = \infty$, we get

$$\ln\left(\lim_{n\to\infty}|C_n^m|\right) = \lim_{n\to\infty}\ln|C_n^m| = -\infty \Rightarrow \lim_{n\to\infty}|C_n^m| = 0.$$

By the Alternating Series Test, $\sum_{n=0}^{\infty} C_n^m x^n$ is convergent.

(c) Before we check the case $m > 0$, we introduce the Raabe's Test:

The Raabe's Test. *Suppose a series* $\sum_{n=1}^{\infty} a_n$ *satisfies*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \quad \textit{and} \quad \lim_{n\to\infty} n\left(\left|\frac{a_{n+1}}{a_n}\right| - 1\right) < -1,$$

then the series is absolutely convergent.

Remark that the p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ satisfies the condition, so the Raabe's Test is a Comparison Test with p-series.

<u>If $m > 0$</u>, then

$$\lim_{n\to\infty} n\left(\left|\frac{a_{n+1}}{a_n}\right| - 1\right) = \lim_{n\to\infty} n\left(\left|\frac{C_{n+1}^m}{C_n^m}\right| - 1\right) = \lim_{n\to\infty} n\left(\frac{\left|\frac{m(m-1)\cdots(m-n)}{n!}\right|}{\left|\frac{m(m-1)\cdots(m-n+1)}{n!}\right|} - 1\right)$$
$$= \lim_{n\to\infty} n\left(\frac{|m-n|}{n+1} - 1\right) = \lim_{n\to\infty} n\left(\frac{n-m}{n+1} - 1\right)$$
$$= -(m+1)\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = -(m+1) < -1.$$

By the Raabe's Test, $\sum_{n=0}^{\infty} C_n^m x^n$ is convergent.

Example 7. We will prove $(1+x)^m = \sum_{n=0}^{\infty} C_n^m x^n$ on $|x| < 1$.

(a) Let $g(x) = \sum_{n=0}^{\infty} C_n^m x^n$ on the interval of convergence $(-1,1)$. We will show that $\underline{(1+x)g'(x) = mg(x)}$ on the interval of convergence $(-1,1)$.

We compute $g'(x) = \sum_{n=1}^{\infty} C_n^m n x^{n-1}$ on the interval of convergence $(-1,1)$, and

$$\begin{aligned}
(1+x)g'(x) &= (1+x)\sum_{n=1}^{\infty} C_n^m n x^{n-1} = \sum_{n=1}^{\infty} C_n^m n x^{n-1} + \sum_{n=1}^{\infty} C_n^m n x^n \\
&= \sum_{n=0}^{\infty} C_{n+1}^m (n+1) x^n + \sum_{n=0}^{\infty} C_n^m n x^n \\
&= \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)(m-n)(n+1)}{(n+1)!} x^n \\
&\quad + \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)n}{n!} x^n \\
&= \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)((m-n)+n)}{n!} x^n \\
&= m\sum_{n=0}^{\infty} C_n^m x^n = mg(x).
\end{aligned}$$

(b) Solve the differential equation $(1+x)g'(x) = mg(x), g(0) = 1, |x| < 1$. It is separable equation, so we have

$$\frac{g'(x)}{g(x)} = \frac{m}{1+x} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(\ln g(x)) = \frac{m}{1+x} \Rightarrow \ln g(x) = m\ln(1+x) + C.$$

Since $g(0) = 1$, we know that $C = 0$. Hence $\ln g(x) = m\ln(1+x) = \ln(1+x)^m$ and it implies $g(x) = \sum_{n=0}^{\infty} C_n^m x^n = (1+x)^m$ on $|x| < 1$.