11.10 Taylor and Maclaurin Series (page 759)

In this section, we will answer two questions: Which functions have power series representation? How can we find such representation?

First, suppose that a smooth function $f(x)$ can be represented by a power series:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots, \qquad \text{if } |x-a| < R. \tag{1}$$

- Put $x = a$, then we get ______.
- Since $f'(x) =$ ____________________________,

  we put $x = a$ and get ______.
- Since $f''(x) =$ ______________, we put $x = a$ and get ______.
- By induction, since $f^{(k)}(x) =$ ______________, we have ______.

Theorem 1 (page 759)**.** *If $f(x)$ has a power series representation (expansion) at a:*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ for } |x-a| < R$$

then its coefficients are given by the formula $c_n = \dfrac{f^{(n)}(a)}{n!}$.

Definition 2 (page 760)**.** Given a smooth function $f(x)$, define the *Taylor series of the function $f(x)$ at a* (or *about a* or *centered at a*) (函數 $f(x)$ 在 $x = a$ 處的泰勒級數) by

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots. \tag{2}$$

For the special cases $a = 0$ the series (2) becomes

$$M(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots.$$

This case the function $M(x)$ is given the special name *Maclaurin series* (馬克勞林級數).

☐ 由前面討論知道:「若 $f(x)$ 可表示成冪級數時」, 則 $f(x)$ 和它的泰勒級數 $T(x)$ 一致。

☐ 我們必須追問 (研究): 有哪些函數「可以」寫成冪級數?(存在函數無法表示成冪級數。)

Example 3 (page 760)**.** Find the Maclaurin series of the function $f(x) = \mathrm{e}^x$ and its radius of convergence.

Solution. Since $f^{(n)}(x) =$ ____, we know that $f^{(n)}(0) =$ __________ for all $n \in \mathbb{N}$ or $n = 0$. Therefore the Maclaurin series of $f(x) = \mathrm{e}^x$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

To find the radius of convergence, we let $a_n =$ ______, then

$$\left|\frac{a_{n+1}}{a_n}\right| =$$

By the ______________, the radius of convergence is _________.

Question 4 (page 761)**.** Under what circumstances is a function equal to the sum of its Taylor series? In other words, if $f(x)$ has derivatives of all orders, when is it true that

$$f(x) \stackrel{?}{=} T(x) \stackrel{\text{def.}}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \stackrel{\text{def.}}{=} \lim_{n\to\infty} T_n(x),$$

where

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \quad (3)$$

Definition 5 (page 761)**.**

(a) The polynomial $T_n(x)$ in (3) is called *n-th degree Taylor polynomial of $f(x)$ at a* ($f(x)$ 在 $x = a$ 的 n-階泰勒多項式).

(b) Define the *remainder* (餘項) of the Taylor series as $r_n(x) \stackrel{\text{def.}}{=} f(x) - T_n(x)$.

Theorem 6 (page 761)**.** *A smooth function $f(x) = T(x)$ on the interval $|x-a| < R$ if and only if $\lim_{n\to\infty} r_n(x) = 0$ for $|x-a| < R$.*

Proof. ($\Rightarrow$) Since $f(x) = \lim_{n\to\infty} T_n(x)$ and $r_x(x) = f(x) - T_n(x)$, we have

$$\lim_{n\to\infty} r_n(x) = \lim_{n\to\infty} (f(x) - T_n(x)) = f(x) - \lim_{n\to\infty} T_n(x) = f(x) - f(x) = 0.$$

($\Leftarrow$) Conversely, since $\lim_{n\to\infty} r_n(x) = 0$ and $T_n(x) = f(x) - r_n(x)$, we have

$$T(x) = \lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} (f(x) - r_n(x)) = f(x) - \lim_{n\to\infty} r_n(x) = f(x) - 0 = f(x).$$

□

☐ 想清楚: 函數是否與其泰勒級數「相同」, 和泰勒級數的「收斂範圍」是兩回事。

☐ 定理得知: 函數與其泰勒級數在其收斂範圍內「相等」的等價條件是「餘項趨近於零」。

Question 7 (page 762)**.** How do we show that $\lim_{n\to\infty} r_n(x) = 0$ for a specific function $f(x)$?

Theorem 8. *Suppose that $f(x)$ has continuous derivative at $x = a$ up to $n+1$ order, then*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + r_n(x) = T_n(x) + r_n(x),$$

where $r_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, c is a number between a and x.

Proof. Without loss of generality, we assume $a < x$. Consider the function

$$g(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n,$$

then $g(t)$ is continuous on $[a, x]$, and

$$\begin{aligned}
g'(t) &= -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}\,k(x-t)^{k-1}(-1) \\
&= -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} \\
&= -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!}(x-t)^k = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.
\end{aligned}$$

Let $h(t) = (x-t)^{n+1}$, by the Cauchy Theorem (generalized Mean Value Theorem), then there exists $c \in (a, x)$ such that

$$\frac{g'(c)}{h'(c)} = \frac{g(x) - g(a)}{h(x) - h(a)} \Rightarrow \frac{-\frac{f^{(n+1)}(c)(x-c)^n}{n!}}{-(n+1)(x-c)^n} = \frac{0 - r_n(x)}{0 - (x-a)^{n+1}},$$

so

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

□

□ 想成是「均值定理」的高階版本, 餘項形式和泰勒多項式一樣, 只是高次微分處代入 c。

Once we have this expression of the remainder, we can estimate it by the following theorem.

Taylor's Inequality (page 762)**.** *If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $r_n(x)$ of the Taylor series satisfies the inequality*

$$|r_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \textit{for} \quad |x-a| \le d.$$

Example 9 (page 763).

(1) Prove that e^x is equal to the sum of Maclaurin series.

(2) Find the Taylor series for $f(x) = e^x$ at $a = 2$.

Solution.

(1) If $f(x) = e^x$, then $f^{(n)}(x) = e^x$ for all $n \in \mathbb{N}$. Given $x \in \mathbb{R}$, there is a positive number d such that $|x| \le d$. Since $|f^{(n+1)}(x)| = e^x \le e^d$, we get

$$|r_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \le \underline{\hspace{6em}} \quad \text{for} \quad |x| \le d.$$

Notice that $\underline{e^d \text{ is a number independent of } n}$, so we have

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} =$$

By the Squeeze Theorem $\lim_{n\to\infty} r_n(x) = 0$, and $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ for all $x \in \mathbb{R}$.

(2) We have $f^{(n)}(2) = e^2$, so the Taylor series for $f(x) = e^x$ at $x = 2$ is

Another viewpoint is ________________________.

Example 10 (page 764). Find the Maclrurin series for $f(x) = \sin x$. Prove that it represents $\sin x$ for all x.

Solution. We compute for $k = 0, 1, 2, \ldots$,

$f^{(4k)}(x) =$	$f^{(4k+1)}(x) =$	$f^{(4k+2)}(x) =$	$f^{(4k+3)}(x) =$
$f^{(4k)}(0) =$	$f^{(4k+1)}(0) =$	$f^{(4k+2)}(0) =$	$f^{(4k+3)}(0) =$

so the Maclaurin series for $f(x) = \sin x$ is

Since $f^{(n+1)}(x)$ is $\pm \sin x$ or $\pm \cos x$, we know that $|f^{(n+1)}(x)| \le 1$ for all $x \in \mathbb{R}$. By Taylor's Inequality:

$$|r_n(x)| =$$

Since $\underline{\lim_{n\to\infty} \hspace{12em}}$, we have $\lim_{n\to\infty} r_n(x) = 0$ for all $x \in \mathbb{R}$ by the Squeeze Theorem. Thus $\sin x$ is equal to the sum of its Maclaurin series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$.

Example 11 (page 764–765)**.**

(1) Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $x = \frac{\pi}{3}$.

(2) Find the Maclaurin series for $\cos x$.

(3) Find the Maclaurin series for $x \cos x$.

Solution. We have for $k = 0, 1, 2, 3, \ldots$

$f^{(4k)}(x) =$	$f^{(4k+1)}(x) =$	$f^{(4k+2)}(x) =$	$f^{(4k+3)}(x) =$
$f^{(4k)}(\frac{\pi}{3}) =$	$f^{(4k+1)}(\frac{\pi}{3}) =$	$f^{(4k+2)}(\frac{\pi}{3}) =$	$f^{(4k+3)}(\frac{\pi}{3}) =$

(1) The Taylor series at $\frac{\pi}{3}$ is

(2) Instead of computing derivatives and substituting in the Maclaurin series for $\cos x$, we can differentiate the Maclaurin series for $\sin x$:

$$\cos x =$$

Since the Maclaurin series for $\sin x$ converges for all x, the differential series for $\cos x$ also converges for all x.

(3) We can multiply the series for $\cos x$ by x:

$$x \cos x =$$

Example 12 (page 766)**.** Find the Maclaurin series for $f(x) = (1 + x)^m$, where m is *any real number.*

Solution.

Therefore the Maclaurin series for $f(x) = (1 + x)^m$ is

Example 13 (page 766)**.** Find the radius of convergence of the *binomial series* (二項式級數, 從上一個例子推得) $\sum_{n=0}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n$.

Solution. If m is a nonnegative integer, then the terms are eventually 0 and so the series is finite. For other values of m, if the n-th term is a_n, then

$$\left|\frac{a_{n+1}}{a_n}\right| =$$

By the ____________, the binomial series converges if _______ and diverges if _______, and the radius of convergence is _______.

The Binomial Series (page 767)**.** *If m is any real number and $|x| < 1$, then*

$$(1+x)^m = \sum_{n=0}^{\infty} C_n^m x^n = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots.$$

The interval of convergence depends on m: $(-1, 1)$ if $m \le -1$; $(-1, 1]$ if $-1 < m < 0$; $[-1, 1]$ if $m > 0$.

☐ 直接估計餘項趨近於零比較麻煩, 有其他的方法證明二項式函數與二項式級數「相同」。

Definition 14 (page 766)**.** Numbers $C_n^m = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$ are called *binomial coefficients* (二項式係數). Remark that $C_0^m \equiv 1$ for all $m \in \mathbb{R}$.

Example 15 (page 767)**.** Find the Maclaurin series for $g(x) = \frac{1}{\sqrt{4-x}}$ and its radius of convergence.

Solution. We rewrite $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} =$$

Using the binomial series with $m =$ ____ and with x replaced by ____, we have

$$\frac{1}{\sqrt{4-x}} =$$

The series converges if _______________, so the radius of convergence is ____.

Important Maclaurin series and their radii of convergence

(1) $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $\qquad R = 1$

(2) $\mathrm{e}^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\qquad R = \infty$

(3) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\qquad R = \infty$

(4) $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ $\qquad R = \infty$

(5) $\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ $\qquad R = 1$

(6) $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\qquad R = 1$

(7) $(1+x)^m = \displaystyle\sum_{n=0}^{\infty} C_n^m x^n = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!} + \cdots$ $\qquad R = 1$

Example 16 (page 768)**.** Find the sum of the series

$$\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \cdots .$$

Solution.

Example 17 (page 769)**.** Evaluate $\displaystyle\lim_{x \to 0} \frac{\mathrm{e}^x - 1 - x}{x^2}$.

Solution. Using the Maclaurin series for e^x, we have

$$\lim_{x \to 0} \frac{\mathrm{e}^x - 1 - x}{x^2} =$$

$$=$$

because power series are continuous functions.

Multiplication and division of power series, page 770

Example 18 (page 770)**.** Find the first three nonzero terms in the Maclaurin series for (1) $e^x \sin x$ and (2) $\tan x$.

Solution.

Example 19 (page 768)**.**

(1) Evaluate $\int e^{-x^2}\, dx$ as an infinite series.

(2) Evaluate $\int_0^1 e^{-x^2}\, dx$ correct to within an error of 0.001.

Solution.

(1) We replace x with $-x^2$ in the series for e^x and get, for all $x \in \mathbb{R}$,

$$e^{-x^2} =$$

We integrate term by term:

$$\int e^{-x^2}\, dx =$$

The series is convergent ___________. because e^{-x^2} is convergent ___________.

(2) We compute

$$\int_0^1 e^{-x^2}\, dx =$$

$$=$$

$$\approx$$

The Alternating Series Estimation Theorem shows that the error is less than

Example (TA) 20. Let $f(x) = \ln(5 - x)$.

(a) Find the power series representation for $f(x)$ at $x = 0$.

(b) Find $f^{(n)}(0)$.

Solution.

Example (TA) 21.

(a) Write down the general terms the MacLaurin series of $\sin x$ and $\sin^{-1} x$.

(b) Find their radii of convergence.

(c) Find $\lim_{x \to 0} \frac{\sin x \cdot \sin^{-1} x - x^2}{x^6}$.

Solution.