11.9 Representations of Functions as Power Series (page 752)

In this section, we learn how to represent certain types of functions as sums of power series. We will see that it is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials.

Example 1 (page 752)**.** Recall that the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \qquad \text{if } \ |x| < 1.$$

We can express the following functions by manipulating geometric series:

(1) $\dfrac{1}{1+x^2} =$

(2) $\dfrac{x}{2+x} =$

Differentiation and Integration of Power Series, page 754

Theorem 2 (page 754)**.** *If the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$*, then the function* $f(x)$ *defined by*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (*and therefore continuous*) *on the interval* $(a-R, a+R)$ *and*

(a) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$

(b) $\int f(x)\,\mathrm{d}x = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\dfrac{(x-a)^{n+1}}{n+1}.$

☐ 「冪級數」和「多項式」一樣, 可以逐項微分、逐項積分, 而且收斂「半徑」不變。
(term-by-term differentiation and integration)

☐ 重新看待定理中的 (a), (b), 對於收斂的冪級數:

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = \sum_{n=0}^{\infty} \dfrac{\mathrm{d}}{\mathrm{d}x}(c_n(x-a)^n)$ 「微分」和「求和、極限」可交換。

(b) $\int\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)\mathrm{d}x = \sum_{n=0}^{\infty}\int c_n(x-a)^n\mathrm{d}x$ 「積分」和「求和、極限」可交換。

☐「收斂半徑」相同不代表「收斂範圍」相同 (端點收斂性會變), 所以端點一律重新檢查。

Example 3 (page 745)**.** Express the following function as a power series and find its interval of convergence.

(1) $f(x) = \dfrac{1}{(1-x)^2}$ (2) $g(x) = \ln(1+x)$ (3) $h(x) = \tan^{-1} x.$

Solution.

Exercise (page 757)**.** Find a power series representation for the following function and determine the interval of convergence.

(1) $f(x) = \frac{3}{x^2-x-2}$. (partial fraction first) (2) $g(x) = \frac{x^2+x}{(1-x)^3}$.

(3) $h(x) = x^2\tan^{-1}(x^3)$.

Exercise.

(a) Find the radius of convergence and the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}}$.

(b) Let $f(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}}$ when the power series is convergent. Evaluate $f^{(3)}(1)$.

Example 4 (page 758)**.** Find the sum of each of the following series.

$$(1)\ \sum_{n=1}^{\infty} nx^n, \quad |x| < 1 \qquad\qquad (2)\ \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Solution.

Exercise (page 756)**.** Find the sum of each of the following series.

$$(1)\ \sum_{n=1}^{\infty} n(n-1)x^n, \quad |x| < 1 \qquad (2)\ \sum_{n=1}^{\infty} \frac{n^2-n}{2^n}, \qquad (3)\ \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Example 5 (page 750)**.** Evaluate $\int \frac{1}{1+x^7}\,\mathrm{d}x$ as a power series and approximate $\int_0^{0.5} \frac{1}{1+x^7}\,\mathrm{d}x$ correct to within 10^{-7}.

Solution. We express the integrand and then integrate term by term:

$$\frac{1}{1+x^7} =$$

$$\int \frac{1}{1+x^7}\,\mathrm{d}x =$$

This series converges for __________, that is ________.

$$\int_0^{0.5} \frac{1}{1+x^7}\,\mathrm{d}x =$$

When we choose $n = 3$, by the Alternating Series Estimation Theorem, the error is smaller than the term with $b_4 = \frac{1}{29\cdot 2^{29}} \approx 6.4 \times 10^{-11}$, so we have

$$\int_0^{0.5} \frac{1}{1+x^7}\,\mathrm{d}x \approx$$

Example (TA) 6. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Find the intervals of convergence for $f(x)$, $f'(x)$, and $f''(x)$.

(這一個題目觀察的重點是在於端點的收斂性, 最後請寫下你的心得。)

Solution.