11.8 Power Series (page 746)

Definition 1 (page 746). A *power series* (冪級數) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ,$$

where x is a variable and the c_n's are constants called the *coefficients* (係數) of the series.

A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

whose *domain* (定義域) is the set of all x for which the series converges.

☐「冪級數」可想成是「多項式」的推廣 – 多了極限的運算。

Example 2 (page 746). If $c_n \equiv 1$, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots ,$$

which converges when ______________ and diverges when __________.

Definition 3 (page 747). A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a *power series in* $(x-a)$ (以 $(x-a)$ 形式的冪級數) or a *power series centered at* a (以 a 爲中心的冪級數) or *power series about* a (關於 a 的冪級數).

☐ 約定 $(x-a)^0 \equiv 1$, 即使 $x = a$ 也是如此。

☐ 任何關於 a 的冪級數, 必在 $x = a$ 收斂, 所以冪級數的定義域非空集合。

Theorem 4 (page 749). *For a given power series* $\sum_{n=0}^{\infty} c_n (x-a)^n$ *there are only three possibilities:*

(a) *The series converges only when* $x = a$.

(b) *The series converges for all* x.

(c) *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$. (注意此定理還不完整, 端點收斂行爲因級數而異。)

Definition 5 (page 749)**.**

(1) The number R in case (c) is called the *radius of convergence* (收斂半徑) of the power series.

(2) By convention, the radius of convergence is $R = 0$ in case (a) and $R = \infty$ in case (b).

(3) The *interval of convergence* (收斂區間) of a power series is the interval that consists of all values of x for which the series converges. When x is an *endpoint* (端點) of the interval, that is, $x = a \pm R$, anything can happen – the interval of convergence could be

$$(a - R, a + R) \qquad (a - R, a + R] \qquad [a - R, a + R) \qquad [a - R, a + R].$$

Example 6 (page 747)**.** Find the interval of the convergence of the following series:

(a) $\displaystyle\sum_{n=0}^{\infty} n!x^n$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ (Bessel function of order 0) (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$.

Solution.

Example (TA) 7.

(a) Evaluate $\lim\limits_{x \to 0} \frac{1-\cos x}{x^2}$.

(b) Evaluate $\lim\limits_{n \to \infty} \frac{1-\cos\left(\frac{1}{n}\right)}{1-\cos\left(\frac{1}{n+1}\right)}$.

(c) Find the interval of convergence of the power series $\sum\limits_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right) x^n$.

Solution.

Exercise. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n!(3^n+1)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)} x^n.$$

Exercise. Consider the power series $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n} x^n$.

(a) Find the radius of the convergence r.

(b) Discuss whether the power series is convergent or divergent at $x = r$ and $x = -r$.

Exercise. Find the interval of convergence of the series $\sum\limits_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) x^n$.