11.6 Absolute Convergence and the Ratio and Root Tests (page 737)

Definition 1 (page 737-738)**.**

(1) A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* (絕對收斂) if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(2) A series $\sum_{n=1}^{\infty} a_n$ is called *conditionally convergent* (條件收斂) if it is convergent but not absolutely convergent.

Example 2 (page 737)**.**

(a) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

(b) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

Example 3. Determine the series $\sum_{n=1}^{\infty} (-1)^n \sin \frac{1}{n}$ is absolutely convergent, conditionally convergent, or divergent.

Solution.

Exercise. Determine the series (a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ and (b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)}$ is absolutely convergent, conditionally convergent, or divergent.

Theorem 4 (page 738)**.** *If a series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, then it is convergent.*

Proof.

□

The Ratio Test (page 739)**.**

(a) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent* (*and therefore convergent*).

(b) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(c) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, *the Ratio Test is inconclusive* (無法確定的)*; that is, no conclusion can be drawn about the convergence or divergence of* $\sum_{n=1}^{\infty} a_n$.

☐ 加上絕對值後, 級數的「行爲」被公比爲 r 的等比級數控制, 其中 $L < r < 1$。

Example 5 (page 740)**.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent. (a) $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ (b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 2^n}{n!}$.

Solution.

☐ 帶有「指數」或「階乘」的級數, 比值法 (Ratio Test) 通常很好用。

☐ 帶有「多項式」、「有理函數」或帶有「三角函數」, 通常用比較判別法。

☐ 比值判別法無法確定的例子: $\sum_{n=1}^{\infty}\frac{1}{n}$ 發散, 而 $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 收斂。

Exercise (page 743)**.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent. (a) $\sum_{n=1}^{\infty}\frac{2^{n^2}}{n!}$ (b) $\sum_{n=1}^{\infty}\frac{(n!)^2}{(5n)!}$.

The Root Test (page 741)**.**

(a) *If* $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent* (*and therefore convergent*).

(b) *If* $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ *or* $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(c) *If* $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, *the Root Test is inconclusive* (無法確定的).

Example 6 (page 741)**.** Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

Solution.

☐ 通常級數型如 $\sum_{n=1}^{\infty} (a_n)^n$ 可考慮用根式法 (Root Test)。

☐ 比值法比根式法重要一些 (11.8 之後)。

Exercise (page 743)**.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent. (a) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ (b) $\sum_{n=1}^{\infty} (\sqrt{n}-\sqrt{n-1})^{2n}$.

Rearrangements, page 742

If we rearrange the order of the terms in a finite sum, then the value of the sum remains unchanged. But it is *not* always the case for an infinite series.

By a *rearrangement* of an infinite series $\sum_{n=1}^{\infty} a_n$ (更序級數) we mean a series obtained by simply changing the order of the terms. Formally, we will write $\sum_{\sigma(n)} a_{\sigma(n)}$ where $\sigma(n)$ is an one-to-one map from the natural number $\mathbb{N}$ to itself. For instance, a rearrangement of $\sum_{\sigma(n)} a_{\sigma(n)}$ could start as follows:

$$a_2 + a_7 + a_3 + a_{32} + a_{15} + a_{10} + a_{200} + \cdots.$$

It turns out that

Theorem 7 (page 742)**.**

(a) *If* $\sum_{n=1}^{\infty} a_n$ *is an absolutely convergent series with sum* s*, then any rearrangement of* $\sum_{n=1}^{\infty} a_n$ *has the same sum* s*.*

(b) *If* $\sum_{n=1}^{\infty} a_n$ *is a conditionally convergent series and* r *is any real number whatsoever, then there is a rearrangement of* $\sum_{n=1}^{\infty} a_n$ *that has a sum equal to* r*.*

Example 8 (page 742)**.** Consider the alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots \qquad (1)$$

If we multiply this series by $\frac{1}{2}$ and insert 0 between the terms of new series, we get

$$\frac{1}{2}S = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots \qquad (2)$$

Now we add the series in (1) and (2) to get

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots \qquad (3)$$

Notice that the series in (3) contains the same terms as in (1).

Theorem 9. *If* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are two absolutely convergent series with sum* A *and* B*, respectively, then the product series* $\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$ *and any rearrangement of* $\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$ *has a sum equal to* AB*.*

Appendix

Proof of Ratio Test, page 739.

(a) Since $L < 1$, we can choose a number r such that $L < r < 1$. Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ and $L < r$, the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ will eventually be less than r; that is, there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \quad \Leftrightarrow \quad |a_{n+1}| < |a_n| r \quad \text{whwnever} \quad n \geq N.$$

In general, we get

$$|a_{N+k}| < |a_{N+k-1}|r < |a_{N+k-2}|r^2 < \cdots < |a_N|r^k \quad \text{for all} \quad k \geq 1.$$

By the Comparison Test, we know

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| < \sum_{k=1}^{\infty} |a_N| r^k = \frac{|a_N|}{1-r}.$$

Hence $\sum_{n=1}^{\infty} |a_n|$ is convergent, and $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ will eventually be greater than 1; that is, there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad \Leftrightarrow \quad |a_{n+1}| > |a_n| \quad \text{whwnever} \quad n \geq N.$$

Since $\lim_{n\to\infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

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