11.5 Alternating Series (page 732)

Definition 1 (page 732)**.** An *alternating series* (交錯級數) is a series whose terms are alternately positive and negative.

Example 2 (page 732)**.** Two examples of alternating series are

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$
$$\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots \qquad .$$

Alternating Series Test (page 727)**.** *If the alternating series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots, \quad \text{where } b_n > 0,$$

satisfies

(a) $b_{n+1} \le b_n$ *for all* n

(b) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

Figure 1: Alternating series test.

☐ 交錯級數只要「從某一項之後遞減」並且「趨近於零」, 則級數收斂。

Example 3 (page 734)**.** Determine whether the following series converges or diverges.

$$\text{(a)} \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} \qquad \text{(b)} \sum_{n=1}^{\infty}\frac{(-1)^{n}3n}{4n-1}.$$

Solution.

Example 4 (page 734)**.** Test the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ for convergence or divergence.

Solution.

Exercise. Test the series $\sum_{n=1}^{\infty}(-1)^{n}\left(e^{\frac{1}{n}}-1\right)$ for convergence or divergence.

Estimating Sums, page 735

Alternating Series Estimation Theorem (page 735)**.** *If* $s=\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *is the sum of an alternating series that satisfies*

(a) $b_{n+1}\leq b_n$

(b) $\lim_{n\to\infty} b_n=0,$

then $|R_n|=|s-s_n|\leq b_{n+1}$.

☐ 「好的」交錯級數 (滿足 (a) 與 (b)), 則級數和與有限項和之誤差只要看第一個忽略項。

☐ 此定理只適用於「交錯級數」, 其他類型的級數不適用。

Example 5 (page 735)**.** Find the sum of the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ correct to three decimal place.

Solution. Since $\frac{1}{(n+1)!}=\frac{1}{(n+1)n!}<\frac{1}{n!}$ and $0\leq\lim_{n\to\infty}\frac{1}{n!}\leq\lim_{n\to\infty}\frac{1}{n}=0$, the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ converges by the ________________________. By the Alternating Series Estimation Theorem we hope $|s-s_n|\leq b_{n+1}<0.0005$, so $(n+1)!>2000$ and $n\geq 6$. Hence $s\approx s_6=\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}=0.368056\ldots\doteq 0.368$ correct to three decimal places with maximum error less than 0.001.

Exercise (page 736)**.** How many terms of the series do we need to add in order to find the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^6}$ correct to four decimal place?