11.4 The Comparison Tests (page 727)

The Comparison Test (page 727). *Suppose that* $\sum\limits_{n=1}^{\infty} a_n$ *and* $\sum\limits_{n=1}^{\infty} b_n$ *are series with positive terms and* $a_n \leq b_n$ *for all* n*.*

(a) *If* $\sum\limits_{n=1}^{\infty} b_n$ *is convergent, then* $\sum\limits_{n=1}^{\infty} a_n$ *is also convergent.*

(b) *If* $\sum\limits_{n=1}^{\infty} a_n$ *is divergent, then* $\sum\limits_{n=1}^{\infty} b_n$ *is also divergent.*

Proof. Let $s_n = \sum\limits_{k=1}^{n} a_k, t_n = \sum\limits_{k=1}^{n} b_k$, and $t = \sum\limits_{k=1}^{\infty} b_k$.

(a) <u>Monotone</u>: Since both series have positive terms, the sequences $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are increasing.

<u>Bounded</u>: Since $a_k \leq b_k$ for all k, we have $s_n \leq t_n \leq t$.

By the ______________________________, $\sum\limits_{n=1}^{\infty} a_n$ converges.

(b) If $\sum\limits_{n=1}^{\infty} a_n$ is divergent, then $s_n \to \infty$, thus $t_n \to \infty$. Therefore $\sum\limits_{n=1}^{\infty} b_n$ diverges.

□

Most of time we use p-series and geometric series for the purpose of comparison.

(1) <u>p-series</u>: $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$. It is convergent if ______ and divergent if ______.

(2) <u>geometric series</u>: $\sum\limits_{n=1}^{\infty} ar^{n-1}$. It is convergent if ______ and divergent if ______.

Example 1. Show that the series $\sum\limits_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent.

Solution.

Exercise. Determine the convergence of the following series:

(a) $\sum\limits_{n=0}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$. (Hint: 分子有理化)

(b) $\sum\limits_{n=1}^{\infty} \sqrt{\sin \frac{1}{n^3}}$. (Hint: ________)

Exercise.

(a) Prove that $\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln n$.

(b) Test for convergence of $\sum_{n=1}^{\infty} \frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{n}}$.

The Limit Comparison Test (page 729)**.** *Suppose that* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with positive terms. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$

where c *is a finite number and* $c > 0$*, then either both series converge or both diverge.*

Proof. Let m and M be positive numbers such that $m < c < M$. Since $\frac{a_n}{b_n}$ is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \Rightarrow mb_n < a_n < Mb_n \qquad \text{when } n > N.$$

By the ____________________, we know both series converge or both diverge. □

□ 比較定理與極限比較定理只適用於「正項級數」。

Example 2 (page 730)**.** Determine whether the following series converges or diverges.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{9^n}{3+10^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}.$$

Solution.

Exercise (page 726)**.** Determine whether the following series converges or diverges.

$$\text{(a)} \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}.$$

Estimating Sums, page 730

If we have used the Comparison Test to show that a series $\sum_{n=1}^{\infty} a_n$ converges by comparison with a series $\sum_{n=1}^{\infty} b_n$, then we may be able to estimate the sum $\sum_{n=1}^{\infty} a_n$ by comparing remainders.

Consider the remainder $R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$ and $T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$. Since $a_n \le b_n$ for all n, we have $R_n \le T_n$.

(1) If $\sum_{n=1}^{\infty} b_n$ is a p-series, we can estimate its remainder T_n as in Section 11.3.

(2) If $\sum_{n=1}^{\infty} b_n$ is a geometric series, we can sum it exactly.

Example 3 (page 730)**.** Use the sum of the first 100 terms to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$. Estimate the error involved in this approximation.

Solution.

Example 4 (page 731)**.** Use $\sum_{n=1}^{10} \frac{\cos^2 n}{5^n} \doteq 0.07393$ to estimate the error of the sum of the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{5^n}$.

Solution.

Exercise. Use $\sum_{n=1}^{10} \frac{1}{3^n+4^n} \doteq 0.19788$ to estimate the error of the sum of the series $\sum_{n=1}^{\infty} \frac{1}{3^n+4^n}$.

Appendix

Example (TA) 5 (page 732)**.**

(a) Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms and $\sum_{n=1}^{\infty} b_n$ is convergent. Show that if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(b) Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms and $\sum_{n=1}^{\infty} b_n$ is divergent. Show that if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

(c) Determine whether the following series converges or diverges.

$$(1)\ \sum_{n=1}^{\infty} \frac{\ln n}{n^3} \qquad (2)\ \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\,\mathrm{e}^n} \qquad (3)\ \sum_{n=2}^{\infty} \frac{1}{\ln n} \qquad (4)\ \sum_{n=1}^{\infty} \frac{\ln n}{n}$$

Solution.