

## 11.3 The Integral Test and Estimates of Sums (page 719)

**The Integral Test** (page 721). Suppose  $f(x)$  is a continuous, positive, decreasing function on  $[1, \infty)$  and let  $a_n = f(n)$ . Then the series  $\sum_{n=1}^{\infty} a_n$  is convergent if and only if the improper integral  $\int_1^{\infty} f(x) dx$  is convergent. In other words,

(a) If  $\int_1^{\infty} f(x) dx$  is convergent, then  $\sum_{n=1}^{\infty} a_n$  is convergent.

(b) If  $\int_1^{\infty} f(x) dx$  is divergent, then  $\sum_{n=1}^{\infty} a_n$  is divergent.

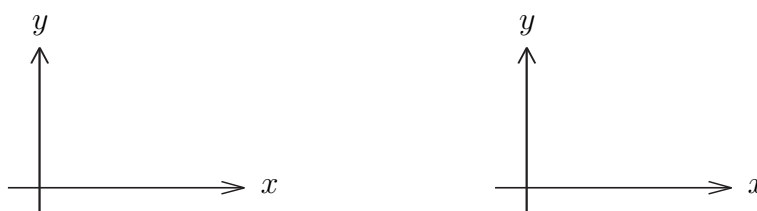


Figure 1: The integral test.

**Theorem 1** (page 721). The  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  ( $p$ -級數) is convergent if  $p > 1$  and divergent if  $p \leq 1$ .

*Proof.* If  $p < 0$ ,

If  $p = 0$ ,

If  $p > 0$ , consider  $f(x) = \frac{1}{x^p}$ , which is continuous, positive and decreasing on  $[1, \infty)$ .  
Since

□

- $f(x)$  必須「恆正」與「遞減」，函數的連續性是要讓積分比較好處理。
- 不見得要「從頭  $n = 1, x = 1$  開始」；收斂和發散和前面有限項無關。
- 定理只是說明積分式與級數有相同的斂散性，不代表兩者具有相同的值。

**Example 2** (page 722). Determine whether the series  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$  converges or diverges.

**Solution.**

□ 先觀察當指標改成  $x$  時有沒有辦法驗證收斂性, 可以的話再逐一檢查條件。

### Estimating the Sum of a Series, page 723

Suppose a series  $\sum_{n=1}^{\infty} a_n$  is convergent by the Integral Test. We can also estimate the size of the *remainder* (餘項)

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots = \sum_{k=n+1}^{\infty} a_k.$$

**Remainder Estimate for the Integral Test** (page 718). Suppose  $f(k) = a_k$ , where  $f(x)$  is a continuous, positive, decreasing function for  $x \geq n$  and  $\sum_{n=1}^{\infty} a_n$  is convergent. If  $R_n = s - s_n$ , then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx. \quad (1)$$

If we add  $s_n$  to each side of the inequalities (1), because  $s_n + R_n = s$ , we get

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq s \leq s_n + \int_n^{\infty} f(x) dx.$$

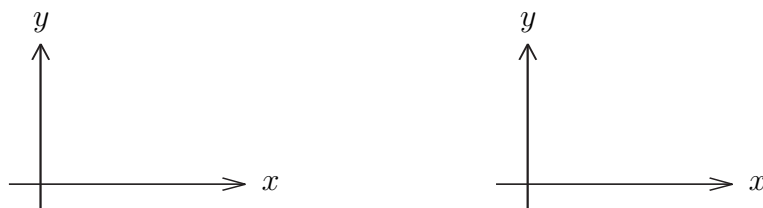


Figure 2: Remainder estimate for the Integral Test.

**Example 3** (page 723). Approximate the sum of the series  $\sum_{n=1}^{\infty} \frac{1}{n^3}$ . How many terms are required to ensure that the sum is accurate to within 0.005?

**Solution.**

**Exercise.** Determine the value of  $p \geq 0$  such that the series  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$  converges.

**Exercise.** Test the series  $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$  where  $p = 1$  and  $p = \frac{3}{2}$ , for convergence.

**Exercise** (page 721). How many terms of the series  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$  would you need to add to find its sum to within 0.01?

## Appendix

**Example (TA) 4.** Determine the values  $p$  such that the series  $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))^p}$  converges.

**Solution.**

**Example (TA) 5** (page 727). Find all values of  $c$  for which the following series converges:

$$\sum_{n=1}^{\infty} \left( \frac{c}{n} - \frac{1}{n+1} \right).$$

**Solution.**

**Example (TA) 6** (page 721). Show that if we want to approximate the sum of the series  $\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$  so that the error is less than 5 in the ninth decimal place, then we need to add more than  $10^{11301}$  terms!

**Solution.**