11.3 The Integral Test and Estimates of Sums (page 719)

The Integral Test (page 721)**.** *Suppose $f(x)$ is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent. In other words,*

(a) *If $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.*

(b) *If $\int_1^{\infty} f(x)\,\mathrm{d}x$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.*

Figure 1: The integral test.

Theorem 1 (page 721)**.** *The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (p-級數) is convergent if $p > 1$ and divergent if $p \leq 1$.*

Proof. If $p < 0$,

If $p = 0$,

If $p > 0$, consider $f(x) = \frac{1}{x^p}$, which is continuous, positive and decreasing on $[1,\infty)$. Since

∎

☐ $f(x)$ 必須「恆正」與「遞減」, 函數的連續性是要讓積分比較好處理。

☐ 不見得要「從頭 $n = 1, x = 1$ 開始」; 收斂和發散和前面有限項無關。

☐ 定理只是說明積分式與級數有相同的斂散性, 不代表兩者具有相同的值。

Example 2 (page 722)**.** Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution.

☐ 先觀察當指標改成 x 時有沒有辦法驗證收斂性, 可以的話再逐一檢查條件。

Estimating the Sum of a Series, page 723

Suppose a series $\sum_{n=1}^{\infty} a_n$ is convergent by the Integral Test. We can also estimate the size of the *remainder* (餘項)

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots = \sum_{k=n+1}^{\infty} a_k.$$

Remainder Estimate for the Integral Test (page 718)**.** *Suppose* $f(k) = a_k$*, where* $f(x)$ *is a continuous, positive, decreasing function for* $x \geq n$ *and* $\sum_{n=1}^{\infty} a_n$ *is convergent. If* $R_n = s - s_n$*, then*

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\,\mathrm{d}x. \tag{1}$$

If we add s_n *to each side of the inequalities* (1)*, because* $s_n + R_n = s$*, we get*

$$s_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq s \leq s_n + \int_{n}^{\infty} f(x)\,\mathrm{d}x.$$

Figure 2: Remainder estimate for the Integral Test.

Example 3 (page 723)**.** Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$. How many terms are required to ensure that the sum is accurate to within 0.005?

Solution.

Exercise. Determine the value of $p \geq 0$ such that the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges.

Exercise. Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ where $p = 1$ and $p = \frac{3}{2}$, for convergence.

Exercise (page 721)**.** How many terms of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ would you need to add to find its sum to within 0.01?

Appendix

Example (TA) 4. Determine the values p such that the series $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))^p}$ converges.

Solution.

Example (TA) 5 (page 727). Find all values of c for which the following series converges:

$$\sum_{n=1}^{\infty}\left(\frac{c}{n}-\frac{1}{n+1}\right).$$

Solution.

Example (TA) 6 (page 721). Show that if we want to approximate the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^{1.001}}$ so that the error is less than 5 in the ninth decimal place, then we need to add more than 10^{11301} terms!

Solution.