11.2 Series (page 707)

Definition 1 (page 707–708)**.** Let $\{a_n\}_{n=1}^{\infty}$ be an infinite sequence.

(1) The *partial sums* (部份和) of the sequence $\{a_n\}_{n=1}^{\infty}$ is defined as

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

These partial sums form a new sequence $\{s_n\}_{n=1}^{\infty}$ (部份和數列).

(2) An *infinite series* (or just a *series* 無窮級數) is denoted by

$$\sum_{n=1}^{\infty} a_n \overset{\text{def.}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n),$$

which means the limit of the partial sums of the sequence $\{a_n\}_{n=1}^{\infty}$.

(3) If the limit $\lim\limits_{n\to\infty} s_n = s$ exists (or convergent) as a finite number, then we say the series $\sum\limits_{n=1}^{\infty} a_n$ *convergent* (收斂), and the number s is called the *sum* of the infinite series $\sum\limits_{n=1}^{\infty} a_n$ (級數和).

(4) If the sequence $\{s_n\}_{n=1}^{\infty}$ is divergent, then the series $\sum\limits_{n=1}^{\infty} a_n$ is called *divergent.*

Example 2 (page 708)**.** In this chapter, we are *not* interested in the infinite *arithmetic series* (等差級數、算數級數):

$$\sum_{n=1}^{\infty}(a + (n-1)d) \overset{\text{def.}}{=} a + (a+d) + (a+2d) + \cdots + (a + (n-1)d) + \cdots,$$

where each term is obtained from the preceding one by adding it by the *common difference* (公差) d. This is because the arithmetic series is convergent if and only if $a = 0$ and $d = 0$.

Example 3 (page 709)**.** The *geometric series* (等比級數、幾何級數) is an infinite series

$$\sum_{n=1}^{\infty} ar^{n-1} \overset{\text{def.}}{=} a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots, \quad a \neq 0.$$

Each term is obtained from the preceding one by multiplying it by the *common ratio* (公比) r. We will discuss the convergence or divergence of the geometric series in the following theorem.

Theorem 4 (page 710)**.** *The geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots, \quad a \neq 0.$$

is convergent if $|r| < 1$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if} \quad |r| < 1.$$

If $|r| \geq 1$*, the geometric series is divergent.*

Proof. □

Exercise (page 711)**.** Discuss the series $\sum_{n=0}^{\infty} x^n \overset{\text{def.}}{=} 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$ for $x \in \mathbb{R}$. If the series is convergent, find the sum of the series.

Example 5. Write the number $0.\overline{142857} = 0.142857142857\ldots$ as a ratio of integers (fraction).

Solution.

Exercise. Write the number $0.\overline{285714}$, $2.3\overline{17}$, and $0.\overline{9}$ as a ratio of integers.

Exercise (page 712)**.**

(a) Show that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

(b) Show that the Euler series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Hint: (a) $\frac{1}{n(n+1)} =$ ____________. (b) For $n > 2$, $\frac{1}{n^2} \leq$ __________

Theorem 6 (page 713)**.** *If the series* $\sum\limits_{n=1}^{\infty} a_n$ *is convergent, then* $\lim\limits_{n\to\infty} a_n = 0$.

Proof.

□

Test for Divergence (page 713)**.** *If* $\lim\limits_{n\to\infty} a_n$ *does not exist or if* $\lim\limits_{n\to\infty} a_n \neq 0$*, then the series* $\sum\limits_{n=1}^{\infty} a_n$ *is divergent.*

Example 7 (page 713)**.** The *harmonic series* (調和級數) is an infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} \stackrel{\text{def.}}{=} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots .$$

Show that it is divergent.

Proof.

□

□ 若 $\lim\limits_{n\to\infty} a_n = 0$, 則級數 $\sum\limits_{n=1}^{\infty} a_n$ 收斂與否仍舊無法判定。

例如: 比較調和級數 $\sum\limits_{n=1}^{\infty} \frac{1}{n}$、等比級數 $\sum\limits_{n=1}^{\infty} ar^{n-1}$ 或歐拉級數 $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$。

Theorem 8 (page 714). *If* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent series, then so are the series* $\sum_{n=1}^{\infty} c\,a_n$ *(where c is a constant),* $\sum_{n=1}^{\infty}(a_n+b_n)$, *and* $\sum_{n=1}^{\infty}(a_n-b_n)$, *and*

(a) $\sum_{n=1}^{\infty} c\,a_n = c\sum_{n=1}^{\infty} a_n$.

(b) $\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

(c) $\sum_{n=1}^{\infty}(a_n-b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

☐ 各別的級數和 $\sum_{n=1}^{\infty} a_n$ 與 $\sum_{n=1}^{\infty} b_n$ 之「收斂」很重要。

☐ 各項相加後得到的新的級數和與各別的級數和再相加相同。

☐ 注意! $\sum_{n=1}^{\infty} a_n b_n \neq \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n$。兩數列相乘的級數和不會等於各別級數和再相乘!

☐ 級數和的收斂與否和前面有限項無關。

☐ 若 $\sum_{n=1}^{\infty} a_n$ 收斂而 $\sum_{n=1}^{\infty} b_n$ 發散, 則 $\sum_{n=1}^{\infty}(a_n+b_n)$ 發散。(習題 11.2, #83。)

☐ 若 $\sum_{n=1}^{\infty} a_n$ 與 $\sum_{n=1}^{\infty} b_n$ 發散, 則 $\sum_{n=1}^{\infty}(a_n+b_n)$ 不一定收斂也不一定發散。(習題 11.2, #84。)

Example (TA) 9 (page 715–716). Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\text{(a)} \sum_{n=1}^{\infty} \sqrt[n]{2} \qquad \text{(b)} \sum_{n=1}^{\infty}\left(\frac{1}{\mathrm{e}^n}+\frac{1}{n(n+1)}\right) \qquad \text{(c)} \sum_{n=2}^{\infty}\frac{1}{n^3-n}.$$

Solution.

Exercise (page 715–716)**.** Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1+2^n}{3^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right) \qquad \text{(c)} \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right).$$

Example (TA) 10 (page 718)**.**

(a) A sequence $\{a_n\}$ is defined recursively by the equation $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for $n \geq 3$, where a_1 and a_2 can be any real numbers. Experiment with various values of a_1 and a_2 and use your calculator to guess the limit of the sequence.

(b) Find $\lim_{n\to\infty} a_n$ in terms of $a_1$and a_2 by expressing $a_{n+1} - a_n$ in terms of $a_2 - a_1$ and summing a series.

Solution.

Exercise (page 719)**.** Consider the series $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

(a) Find the partial sums s_1, s_2, s_3, an s_4. Do you recognize the denominators? Use the patten to guess a formula for s_n.

(b) Use mathematical induction to prove your guess.

(c) Show that the given infinite series is convergent, and find its sum.