Chapter 11 Infinite Sequences and Series

11.1 Sequences (page 694)

Definition 1 (page 694)**.**

(1) A *sequence* (數列) is a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots.$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *n-th term*.

(2) An *infinite sequence* (無窮數列) is a sequence that each term a_n has a successor a_{n+1}.

(3) The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Example 2 (page 694)**.** Some sequences can be defined by giving a formula for the n-th term. There are three methods to describe a sequence. Notice that n doesn't have to start at 1.

(a) $\{\frac{n}{n+1}\}_{n=1}^{\infty}$, $a_n = \frac{n}{n+1}$, $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\}$.

(b) $\{\frac{(-1)^n(n+1)}{3^n}\}_{n=1}^{\infty}$, $a_n = \frac{(-1)^n(n+1)}{3^n}$, $\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \ldots, \frac{(-1)^n(n+1)}{3^n}, \ldots\}$.

(c) $\{\sqrt{n-3}\}_{n=3}^{\infty}$, $a_n = \sqrt{n-3}, n \geq 3$, $\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\}$.

(d) $\{\cos\frac{n\pi}{6}\}_{n=0}^{\infty}$, $a_n = \cos\frac{n\pi}{6}, n \geq 0$, $\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos\frac{n\pi}{6}, \ldots\}$.

Example 3 (page 695)**.** Here are some sequences that don't have a simple defining equation.

(a) The *Fibonacci sequence* (費波那契數列) $\{f_n\}$ is defined recursively by the conditions

$$f_1 = f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

The first few terms are $\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$. This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits.

(b) If we let a_n be the digit in the n-th decimal place of the number $\sqrt{2}$, then $\{a_n\}$ is a well-defined sequence whose first few terms are $\{4, 1, 4, 2, 1, 3, 5, 6, 2, \ldots\}$.

Definition 4 (page 696)**.** (數列極限之收斂或發散)

(1) A sequence $\{a_n\}$ has the *limit* L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \ \text{ as } \ n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

(2) If $\lim_{n\to\infty} a_n$ exists, we say the sequence *converges* (or is *convergent*, 收斂). Otherwise, we say the sequence *diverges* (or is *divergent*, 發散).

(3) If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$.

Theorem 5. *If $\lim_{n\to\infty} a_n$ exists, then it is unique.*

Property 6 (Limit Laws for Sequences, page 697)**.** *If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then*

(1) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

(2) $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$.

(3) $\lim_{n\to\infty} c\,a_n = c\lim_{n\to\infty} a_n$. *In particular,* $\lim_{n\to\infty} c = c$.

(4) $\lim_{n\to\infty}(a_nb_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.

(5) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ *if* $\lim_{n\to\infty} b_n \neq 0$.

(6) $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ *if* $p > 0$ *and* $a_n > 0$.

The Squeeze Theorem (夾擠定理, page 698)**.** *If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Theorem 7. *If $\lim_{n\to\infty} a_n = L$, then the limit of any subsequences $\lim_{k\to\infty} a_{n_k} = L$.*

☐ 極限若存在, 真相只有一個!

☐ 數列的極限與函數的極限一樣有「四則運算」以及「夾擠定理」。

☐ 夾擠定理, 只要確定某一項之後三個數列有大小關係即可, 和前面有限項的大小無關。

☐ 子數列存在性定理一般的應用是考慮其否逆命題 – 證明原數列極限不存在。

Theorem 8 (page 698)**.** *If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Since ________________, by the ________________, we have $\lim_{n\to\infty} a_n = 0$. ☐

Theorem 9 (page 697)**.** *If* $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$ *when* n *is an integer, then* $\lim_{n\to\infty} a_n = L$*.*

Theorem 10 (page 699)**.** *If* $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f(L).$$

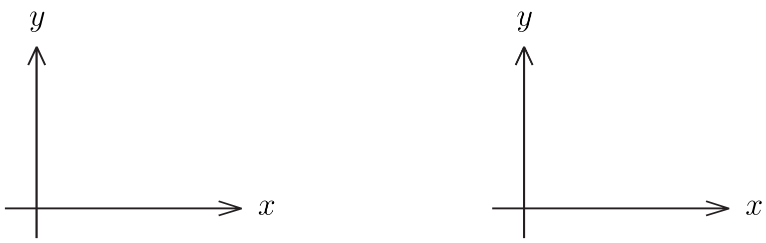

Figure 1: Relations between functions and sequences.

☐ 有了定理 9、定理 10, 就可以將上學期學過函數的極限「應用」到數列的極限, 超好用!

☐ 定理 10 意義:「連續函數」才可以和數列的「極限」交換順序。

☐ 若 $\lim_{n\to\infty} a_n = 0$, 則 $\lim_{n\to\infty} |a_n| = \left|\lim_{n\to\infty} a_n\right| = 0$。(因爲絕對值函數爲連續函數)

Example 11. Discuss the convergence or divergence of the following sequences:
(a) $a_n = \frac{-n^2+1}{2n^2+3n}$ (b) $b_n = \frac{n!}{n^n}$ (c) $c_n = \frac{(-1)^n}{n}$ (d) $d_n = \frac{\ln n}{n}$ (e) $e_n = \sin(\frac{\pi}{n})$.

Solution.

Exercise (page 704)**.** Determine whether the sequence converges or diverges. If it converges, find the limit. (a) $a_n = \frac{n^2}{\sqrt{n^3+4n}}$ (b) $b_n = \frac{(2n-1)!}{(2n+1)!}$ (c) $c_n = \frac{\cos^2 n}{2^n}$ (d) $d_n = \left(1+\frac{2}{n}\right)^n$ (e) $e_n = n - \sqrt{n+1}\sqrt{n+3}$.

Theorem 12 (page 700)**.** *The sequence* $\{r^n\}_{n=1}^{\infty}$ *is convergent if* $-1 < r \le 1$ *and divergent for all other values of* r*. Furthermore, we have*

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1. \end{cases}$$

Proof. Consider $f(x) = a^x$. We know $\lim_{x\to\infty} a^x = \infty$ if $a > 1$; $\lim_{x\to\infty} a^x = 0$ if $0 < a < 1$.

(1) Let $a = r$, we get

(2) If $\underline{r = 1}$,

(3) If $\underline{r = 0}$,

(4) If $\underline{-1 < r < 0}$,

(5) If $\underline{r = -1}$,

(6) If $\underline{r < -1}$,

□

Exercise. Show that $\lim_{n\to\infty} nr^n = 0$ if $|r| < 1$.

Definition 13 (page 700)**.** A sequence $\{a_n\}$ is called *increasing* (遞增) if $a_n < a_{n+1}$ for all $n \ge 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called *decreasing* (遞減) if $a_n > a_{n+1}$ for all $n \ge 1$. A sequence is *monotonic* (單調) if it is either increasing or decreasing.

Definition 14 (page 701)**.** A sequence $\{a_n\}$ is *bounded above* (有上界) if there is a number M such that $a_n \le M$ for all $n \ge 1$. It is *bounded below* (有下界) if there is a number m such that $m \le a_n$ for all $n \ge 1$. If it is bounded above and below, then $\{a_n\}$ is a *bounded sequence* (有界數列).

Monotonic Sequence Theorem (page 702)**.** *Every bounded, monotonic sequence is convergent.* (單調有界數列必收斂。)

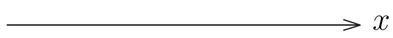

Figure 2: Monotonic sequence theorem.

□ 有界數列未必收斂, 例如: ________________________。

□ 單調數列未必收斂, 例如: ________________________。

□ 定理證明要用到實數的完備性公設 (completeness axiom)。

Example 15 (page 703)**.** Investigate the sequence $\{a_n\}_{n=1}^{\infty}$ defined by the *recurrence relation* (遞迴關係): $a_1 = 2, a_{n+1} = \frac{1}{2}(a_n + 6)$ for $n = 1, 2, 3, \ldots$.

Solution. Monotone: We claim: $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

(1) When $\underline{n = 1}$,

(2) Assume that it is true for $\underline{n = k}$, that is, $a_{k+1} > a_k$.

(3) When $\underline{n = k + 1}$,

(4) By ________________________, we know $\{a_n\}$ is monotone.

Bounded: We claim: $a_n < 6$ for all $n \in \mathbb{N}$.

(1) When $\underline{n = 1}$,

(2) Assume that it is true for $\underline{n = k}$, that is, $a_k < 6$.

(3) When $\underline{n = k + 1}$,

(4) By ________________________, we know $\{a_n\}$ is bounded above by 6.

Limit: By ______________________________, we know $\lim_{n\to\infty} a_n$ exists. Let $\lim_{n\to\infty} a_n = L$. Since

Example (TA) 16 (page 705)**.** A sequence $\{a_n\}_{n=1}^{\infty}$ is given by $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2 + a_n}$. Show that $\{a_n\}$ is increasing, bounded above by 3, $\lim_{n\to\infty} a_n$ exists, and find $\lim_{n\to\infty} a_n$.

Solution.

Exercise (page 705)**.** Show that the sequence defined by $a_1 = 1, a_{n+1} = 3 - \frac{1}{a_n}$ is increasing and $a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

Example (TA) 17 (page 706)**.** Let $a_n = \left(1 + \frac{1}{n}\right)^n$. Show that $\lim_{n\to\infty} a_n$ exists.

Solution.

Exercise (page 706)**.** Let a and b be positive numbers with $a > b$. Let a_1 be their arithmetic mean (算術平均) and b_1 their geometric mean (幾何平均):

$$a_1 = \frac{a+b}{2}, \qquad b_1 = \sqrt{ab}.$$

Repeat this process so that, in general

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}.$$

(a) Use mathematical induction (數學歸納法) to show that $a_n > a_{n+1} > b_{n+1} > b_n$.

(b) Deduce that both $\{a_n\}$ and $\{b_n\}$ are convergent.

(c) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Gauss called the common value of these limits the *arithmetic-geometric mean* of the numbers a and b.