10.4 Areas and Length in Polar Coordinates (page 669)

In this section, we develop the formula for the area of the region whose boundary is given by a polar equation.

Example 1 (page 669)**.** The area of a sector of a circle with the radius r and the radian θ is $A =$ ________.

Example 2 (page 669)**.** Find the area of a region $\mathcal{R}$ bounded by the polar curve $r = r(\theta)$ and by rays $\theta = a$ and $\theta = b$, where $r(\theta)$ is a positive continuous function and $0 < b - a \leq 2\pi$.

Solution.

Example 3 (page 670)**.** Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution.

Example 4 (page 671)**.** Find the area of the region $\mathcal{R}$ bounded by curves with polar equations $r = f(\theta), r = g(\theta), \theta = a$, and $\theta = b$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < b - a \leq 2\pi$.

Solution.

To find *all* points of intersection of two polar curves, it is recommended that you draw the graphs of both curves.

Example 5 (page 671)**.** Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

Solution.

Arc Length, page 671

To find the length of a polar curve $r = r(\theta), a \leq \theta \leq b$, we regard θ as a parameter and write the parameter equations of the curves as

$$\begin{cases} x = r\cos\theta = r(\theta)\cos\theta \\ y = r\sin\theta = r(\theta)\sin\theta. \end{cases}$$

Then using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} =$$
$$\frac{\mathrm{d}y}{\mathrm{d}\theta} =$$

so

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= (r'(\theta))^2\cos^2\theta - 2r\cdot(r'(\theta))^2\cos\theta\sin\theta + r^2\sin^2\theta \\ &\quad + (r'(\theta))^2\sin^2\theta + 2r\cdot(r'(\theta))^2\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \end{aligned}$$

Assuming that $f'(\theta)$ is continuous, we can write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta =$$

Example 6 (page 672)**.** Find the length of the cardioid $r = 1 + \sin\theta$.

Solution.

Surface Area, page 674

The area of the surface generated by rotating the polar curve $r = f(\theta), a \leq \theta \leq b$ (where $f'(\theta)$ is continuous and $0 \leq a < b \leq \pi$) about the polar axis is

Surface area =

The area of the surface generated by rotating the polar curve $r = f(\theta), a \leq \theta \leq b$ (where $f'(\theta)$ is continuous and $0 \leq a < b \leq \pi$) about the line $\theta = \frac{\pi}{2}$ is

Surface area =

Volume

See section 6.2 and 6.3.