10.3 Polar Coordinates (page 658)

A coordinate system represents a point in the plane by an ordered pair of numbers. We usually use Cartesian coordinates (笛卡爾坐標, 直角坐標), which are directed distances from two perpendicular axes. Here we describe another coordinate system introduced by Newton, called the *polar coordinate system* (極坐標).

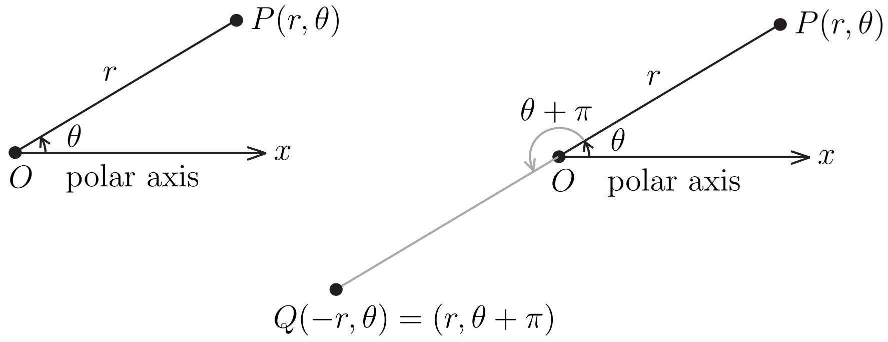

Figure 1: Polar coordinate system. We choose a point in the plane that is called the *pole* and is labeled O. Then we draw a ray starting at O called *polar axis*, which is usually corresponds to the positive x-axis in Cartesian coordinates.

Here are some remarks about the polar coordinate system.

- If P is any other point in the plane, let r be the distance from O to P and let θ be the angle between the polar axis and the line OP. Then the point P is represented by the ordered pair (r,θ) are called *polar coordinates* of P.
- We use the convention that an angle is positive if measured in the counter-clockwise direction from the polar axis and negative in the clockwise direction. If $P=O$, then $r=0$ and we agree that $(0,\theta)$ represents the pole for any value of θ.
- The points $(-r,\theta)$ an (r,θ) lie on the same line through O and at the same direction $|r|$ from O, but on opposite sides of O. If $r>0$, the point (r,θ) lies in the same quadrant as θ; if $r<0$, it lies in the quadrant on the opposite side of the pole.
- Notice that $(-r,\theta)$ represents the same point as $(r,\theta+\pi)$.
- The connection between polar and Cartesian coordinates:

  (a) $\cos\theta=\dfrac{x}{r},\quad \sin\theta=\dfrac{y}{r}.$ (角度和直角坐標與半徑的關係)

  (b) $x=r\cos\theta,\quad y=r\sin\theta.$ (直角坐標用極坐標表達)

  (c) $r^2=x^2+y^2,\quad \tan\theta=\dfrac{y}{x}.$ (極坐標可以用直角坐標表達)

Polar Curves, page 660

The *graph of a polar equation* $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one representation (r, θ) whose coordinates satisfy the equation.

Example 1 (page 660-662)**.** Plot the following curves represented by the polar equation and find a Cartesian equation for this curve.

(a) $r = 2$. ____________________

(b) $\theta = \frac{\pi}{4}$. ____________________

(c) $r = 2\cos\theta$. ____________________

(d) $r = 1 + \sin\theta$. ____________________

(e) $r = \cos 2\theta$. ____________________

Solution.

Symmetry, page 663

When we sketch polar curves it is sometimes helpful to take advantage of symmetry.

(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about ____________________.

(b) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about ____________________.

(c) If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about ____________________.

Tangents to Polar Curves, page 663

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta. \end{cases}$$

Using the method for finding slopes of parametric curves and the Product Rule, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\hspace{4cm}} \qquad (1)$$

- Horizontal tangents: __________________________ (provided that $\frac{\mathrm{d}x}{\mathrm{d}\theta} \neq 0$).
- Vertical tangents: __________________________ (provided that $\frac{\mathrm{d}y}{\mathrm{d}\theta} \neq 0$).
- Tangent lines at the pole: we put $r = 0$ into formula (1) and get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\hspace{5cm}}$$

Example 2 (page 664)**.**

(a) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

Solution.

Graphing Polar Curves with graphing Devices

We can use graphing devices to sketch complicated curves. The curves shown in Figure 2 are almost impossible to produce by hand.

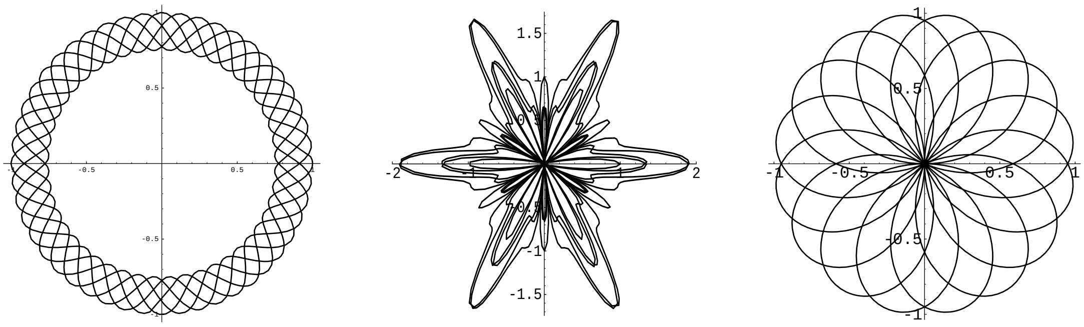

Figure 2: (a) $r = \sin^2(2.4\theta) + \cos^4(2.4\theta), \theta \in [\frac{-2\pi}{2.4}, \frac{22\pi}{2.4}]$. (b) $r = \sin^2(1.2\theta) + \cos^3(6\theta), \theta \in [0, 6\pi]$ (c) $r = \sin\left(\frac{8}{5}\theta\right), \theta \in [0, 10\pi]$.

Some interesting curves and their polar equations.

(a) $r = a\sin(b\theta)$: rose or rhodonea curve (玫瑰線).

(b) $r = a + b\theta$: Archimedean spiral (阿基米德螺線; 等速螺線).

(c) $r = ae^{b\theta}$: logarithmic spiral (對數螺線).

(d) $r^2 = \sin 2\theta$: lemniscate (雙紐線).

(e) $r = e^{\sin\theta} - 2\cos(4\theta)$: butterfly curve (蝶形線).

(f) $r = 1 + c\sin\theta$: limacons de Pascal. (帕斯卡蝸線).

(g) $r = 1 + 2\sin(\frac{\theta}{2})$: nephroid of Freeth.

(h) $r = \sqrt{1 - 0.8\sin^2\theta}$: hippopede.

(i) $r = |\tan\theta|^{|\cot\theta|}$: Valentine curve.

Exercise (page 663)**.** Find a formula for the distance between the points with polar coordinates (r_1, θ_1) and (r_2, θ_2).

Exercise (page 664)**.** Find the slope of the tangent line to the given polar curve at the point specified by the value of θ. (a) $r = \frac{1}{\theta}, \theta = \pi$. (b) $r = \cos 2\theta, \theta = \frac{\pi}{4}$.

Exercise (page 664)**.** Find the points on the given curve the tangent line is horizontal or vertical. (a) $r = 3\cos\theta$. (b) $r = 1 - \sin\theta$.