10.2 Calculus with Parametric Curves (page 649)

We now apply the methods of calculus to parametric curves. In particular, we solve problem involving tangents, area, arc length, surface area, and volume.

Tangents, page 649

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the curve $x = f(t), y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow \underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \quad \text{if} \quad \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0.}$$

We can compute the second derivative $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ as follows:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3}$$

☐ 特別注意: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq \dfrac{\frac{\mathrm{d}^2y}{\mathrm{d}t^2}}{\frac{\mathrm{d}^2x}{\mathrm{d}t^2}}$。

Example 1 (page 650)**.**

(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$ at the point where $\theta = \frac{\pi}{3}$.

(b) At what points its tangents horizontal? When is it vertical?

Solution.

Areas, page 651

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,\mathrm{d}x$,where $F(x) \geq 0$. If the curve is traced out once by the parameter equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\,\mathrm{d}x = \int_\alpha^\beta g(t)f'(t)\,\mathrm{d}t \quad \text{or} \quad \int_\beta^\alpha g(t)f'(t)\,\mathrm{d}t.$$

Example 2 (page 651)**.** Find the area under one arch of the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$.

Solution.

Arc Length, page 652

We already know how to find the length L of a curve C given in the form $y = F(x), a \leq x \leq b$. If $F'(x)$ is continuous, then

$$L = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $\frac{\mathrm{d}x}{\mathrm{d}t} = f'(t) > 0$. This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a, f(\beta) = b$. Then we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_\alpha^\beta \sqrt{1 + \left(\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}\right)^2}\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

The above formula is generally true even if C can't expressed in the form $y = F(x)$.

Theorem 3 (page 649)**.** *If a curve C is described by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is*

$$L = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

Example 4 (page 653)**.** Find the length of one arch of the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$.

Solution.

Surface Area, page 654

In the same way as for arc length, we can obtain a formula for surface area. If the curve given by the parametric equations $x = f(t), y = g(t), \alpha \le t \le \beta$, is rotated about the x-axis, where f', g' are continuous and $g(t) \ge 0$, then the area of the resulting surface is given by

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t.$$

The general symbolic formulas $S = \int 2\pi y \, \mathrm{d}s$ and $S = \int 2\pi x \, \mathrm{d}s$ are still valid, but for parametric curves we use

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t.$$

Example 5 (page 654)**.** Show that the surface area of a sphere of radius r is $4\pi r^2$.

Solution.

Volume

See section 6.2 and 6.3.