Chapter 10 Parametric Equations and Polar Coordinates

10.1 Curves defined by parametric equations (page 640)

Definition 1 (page 640)**.** Suppose that x and y are both given as functions of a third variable t (called a *parameter*, 參數) by the equations

$$x = f(t), \quad y = g(t),$$

(called *parameter equations*, 參數方程). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a *parametric curve* (參數曲線).

Sometimes t can be realized as "time" and we can interpret $(x, y) = (f(t), g(t))$ as the position of a particle at time t, but in many cases, t does *not* necessarily represent time, it is just a variable.

Example 2. How do we express the following curves by parametric equations?

Curve	Parametric Equation
Straight line passing through (x_0, y_0)	$\begin{cases} x = \\ y = \end{cases}$
Circle with center (x_0, y_0) and radius r	$\begin{cases} x = \\ y = \end{cases}$
Ellipse $\dfrac{(x-x_0)^2}{a^2} + \dfrac{(y-y_0)^2}{b^2} = 1$	$\begin{cases} x = \\ y = \end{cases}$
Parabola $(x-x_0)^2 = 4p(y-y_0)$	$\begin{cases} x = \\ y = \end{cases}$
Hyperbola $\dfrac{(x-x_0)^2}{a^2} - \dfrac{(y-y_0)^2}{b^2} = 1$	$\begin{cases} x = \\ y = \end{cases}$

Example 3. Compare the following parametric equations:

(a) $(x, y) = (\cos t, \sin t), 0 \le t \le 2\pi$.

(b) $(x, y) = (\cos t, \sin t), 0 \le t \le 4\pi$.

(c) $(x, y) = (\cos 2t, \sin 2t), 0 \le t \le \pi$.

(d) $(x, y) = (\sin t, \cos t), 0 \le t \le 2\pi$.

(e) $(x, y) = (\cos t, -\sin t), 0 \le t \le 2\pi$.

Example 4 (page 642)**.** Sketch the curve $x = \sin t, y = \sin^2 t$.

Solution.

The Cycloid, page 643

Example 5 (page 643)**.** The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a *cycloid* (擺線). See Figure 1.

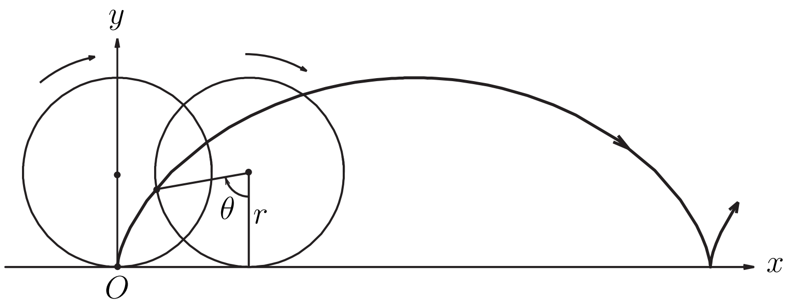

Figure 1: The cycloid.

If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.

Solution.

There are many interesting problems related to cycloids.

Brachistochrone Problem (最速降線問題), page 644

Find the curve along which a particle will slide in the shortest time (under the influence of gravity) from a point A to a lower point B not directly beneath A.

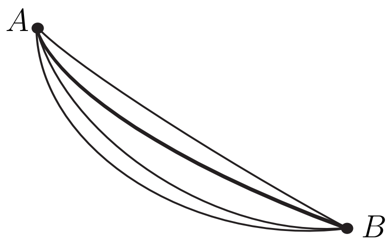

Figure 2: Brachistochrone Problem.

The Swiss mathematician John Bernoulli, who posed this problem in 1696, showed that among all possible curves that join A to B, the particle will take the least time sliding from A to B if the curve is part of an inverted arch of a cycloid.

Tautochrone Problem (等時降線), page 644

The Dutch physicist Huygens had already shown that the cycloid is also the solution to the tautochrone problem: no matter where a particle P is placed on an inverted cycloid, it takes the same time to slide to the bottom.

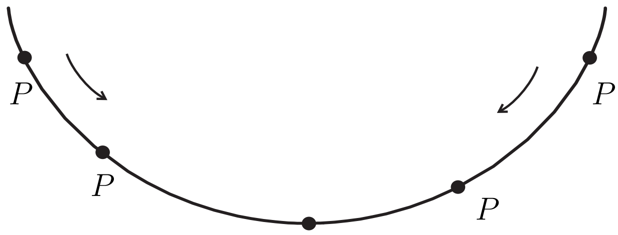

Figure 3: Tautochrone Problem.

Huygens proposed that pendulum clocks should swing in cycloidal arcs because then the pendulum would take the same time to make a complete oscillation whether it swings through a wide or a small arc.

Graphing Devices, page 644

We can use graphing devices to sketch complicated curves. The curves shown in Figure 4 are almost impossible to produce by hand.

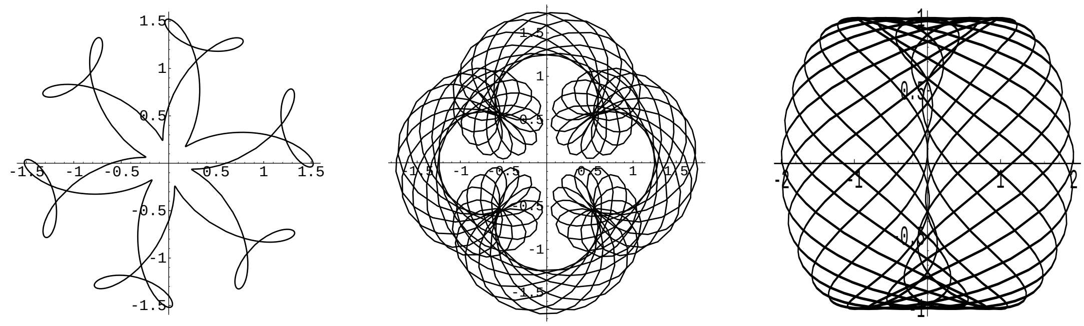

Figure 4: (a) $x = \sin t + \frac{1}{2}\cos 5t + \frac{1}{4}\sin 13t, y = \cos t + \frac{1}{2}\sin 5t + \frac{1}{4}\cos 13t, t \in [0, 2\pi]$.
(b) $x = \sin t + \frac{1}{2}\sin 5t + \frac{1}{4}\cos 2.3t, y = \cos t + \frac{1}{2}\cos 5t + \frac{1}{4}\sin 2.3t, t \in [0, 20\pi]$.
(c) $x = \sin t - \sin 2.3t, y = \cos t, t \in [0, 20\pi]$.

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