9.5 Linear Equations (page 620)

Definition 1 (page 620)**.** A first-order *linear differential equation* (一階線性微分方程) is one that can be put into the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x), \tag{1}$$

where $P(x)$ and $Q(x)$ are continuous functions on a given interval.

Every first-order linear differential equation can be solved by the method of *integrating factor* (積分因子法). The idea comes from the Product Rule of the derivative. Suppose $I(x)$ is a differentiable function. We multiply $I(x)$ on both sides of (1):

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)P(x)y = I(x)Q(x), \tag{2}$$

We hope left hand side of equation (2) is the derivative of the product $I(x)y(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(I(x)y(x)) = I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}I}{\mathrm{d}x}y. \tag{3}$$

Suppose that such $I(x)$ exists, then we can solve the equation (1) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}(I(x)y(x)) = I(x)Q(x) \Rightarrow$$

Now we prove the existence of $I(x)$. Equation (3) gives

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}I}{\mathrm{d}x}y = I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)P(x)y \Rightarrow \frac{\mathrm{d}I}{\mathrm{d}x} = I(x)P(x) \quad \text{for} \quad y \neq 0.$$

This is a separable differential equation for I, so we can solve it:

$$\frac{1}{I(x)}\frac{\mathrm{d}I}{\mathrm{d}x} = P(x) \Rightarrow$$

Hence one solution is $I(x) = \mathrm{e}^{\int P(x)\,\mathrm{d}x}$.

Conclusion (page 621)**.** To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the *integrating factor* (積分因子) $I(x) = \mathrm{e}^{\int P(x)\,\mathrm{d}x}$ and integrate both sides.

☐ 務必要先將一次微分前的係數變成 1, 再用積分因子法。

Example 2. Solve the initial value problem:

$$\begin{cases} y' + y\sec^2 x = x\mathrm{e}^{-\tan x} \\ y(0) = \pi \end{cases}.$$

Solution.

Example 3.

(a) Solve the initial value problem:

$$\begin{cases} x(x+3)y' + \left(2x + \dfrac{3}{2}\right) y = x^{\frac{1}{2}}(x+3)^{\frac{1}{2}}, \\ y(1) = \dfrac{1}{2}, \quad x > 0. \end{cases}$$

(b) Discuss the asymptotic behavior $\lim_{x\to\infty} y(x)$ and $\lim_{x\to 0^+} y(x)$.

Solution.

Application to Electric Circuits, page 599, 623.

The simple electric circuit shown in Figure 1 contains an electromotive force (電動勢) (usually a battery or generator) that produces a voltage of $E(t)$ volts (V) (電壓; 單位: 伏特) and a current of $I(t)$ amperes (A) (電流; 單位: 安培) at time t. The circuit also contains a resistor with a resistance of R ohms (Ω) (電阻; 單位: 歐姆) and an inductor with an inductance of L henries (H) (誘導器; 單位: 亨利). Ohm's

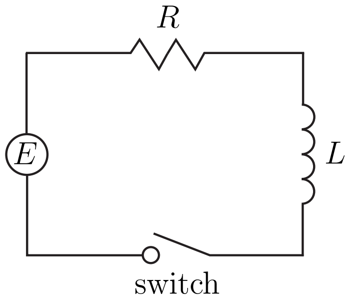

Figure 1: Electric circuits.

Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L\,\frac{\mathrm{d}I}{\mathrm{d}t}$. One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage $E(t)$. Thus we have

$$L\,\frac{\mathrm{d}I}{\mathrm{d}t} + RI = E(t),$$

which is a first-order differential equation that models the current I at time t.

Example 4 (page 623)**.** Suppose that the resistance is $12\,\Omega$ and the inductance is $4\,\mathrm{H}$. If a battery gives a constant voltage of $E(t) = 60\sin 30t\,\mathrm{V}$ and the switch is closed when $t = 0$. Find $I(t)$.

Solution.