9.3 Separable Equations (page 599)

In this section, we examine a certain type of differential equation that can be solved explicitly.

Definition 1 (page 599)**.** A *separable equation* (分離變數型微分方程) is a first-order differential equation in which the expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be factored as a function of x times a function of y. In other words, it can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)f(y) \Rightarrow \frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \quad \text{if } f(y) \neq 0.$$

To solve this equation, we integrate both sides of the equations:

$$\int \frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int g(x)\,\mathrm{d}x \Rightarrow \int \frac{1}{f(y)}\,\mathrm{d}y = \int g(x)\,\mathrm{d}x.$$

Example 2. Solve the initial value problem:

$$\begin{cases} (xy^2 + y^2 + x + 1)\,\mathrm{d}x + (y-1)\,\mathrm{d}y = 0, \\ y(2) = 0 \end{cases}.$$

Solution.

Example 3 (page 605)**.** Solve the differential equation $xy' = y + x\,\mathrm{e}^{\frac{y}{x}}$ by making the change of variable $v = \frac{y}{x}$.

Solution.

Orthogonal Trajectories, page 603

Definition 4 (page 603)**.** An *orthogonal trajectory* (正交軌線) of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles.

In section 3.5, **Example 6**, we have list many examples:

- $x^2 + y^2 = r^2$, $ax + by = 0$.
- $x^2 + y^2 = ax$, $x^2 + y^2 = by$.
- $y = cx^2$, $x^2 + 2y^2 = k$.
- $y = ax^3$, $x^2 + 3y^2 = b$.

Given a family of curves, we can get an orthogonal trajectory of the family by solving the differential equation.

Example 5 (page 597)**.** Show that the orthogonal trajectories of the family of curves $x^2 + y^2 = r^2$ is $ax + by = 0$.

Solution. First, we find the slope of tangent line of $x^2 + y^2 = r^2$:

We want to find family of curve $y(x)$ satisfies

Hence the family of curves is line passing through the origin and $ax + by = 0$ is another expression.

Example 6. Show that the orthogonal trajectories of the family of curves $x^2 + y^2 = ax$ is $x^2 + y^2 = by$.

Solution.

Example 7. Show that the orthogonal trajectories of the family of curves $x^2+y^2 = ax$ is $x^2+y^2 = by$.

Solution. We implicit differentiate $x^2+y^2 = ax$ with respect to x and get $2x+2yy' = a$, so

$$y' = \frac{1}{2y}(a-2x) = \frac{1}{2y}\left(\frac{x^2+y^2}{x} - 2x\right) = \frac{y^2-x^2}{2xy}.$$

We want to find a family of curves $(x, y(x))$ satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{x^2-y^2} = \frac{2\cdot\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2}.$$

Let $y = v(x)x$, then $y' = v'x + v$ and the differential equation becomes

$$xv' + v = \frac{2v}{1-v^2} \Rightarrow xv' = \frac{2v}{1-v^2} - v = \frac{2v - v + v^3}{1-v^2} = \frac{v(v^2+1)}{1-v^2}.$$

So we have

$$\frac{1-v^2}{v(v^2+1)}v' = \frac{1}{x} \Rightarrow \int \frac{1-v^2}{v(v^2+1)}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$$

We deal with the left hand side integration

$$\begin{aligned}\int \frac{1-v^2}{v(v^2+1)}\,\mathrm{d}v &= \int \left(\frac{1}{v} + \frac{-2v}{v^2+1}\right)\mathrm{d}v = \ln|v| - \ln|v^2+1| = \ln\left|\frac{v}{v^2+1}\right| \\ &= \ln\left|\frac{\frac{y}{x}}{\left(\frac{y}{x}\right)^2+1}\right| = \ln\left|\frac{xy}{x^2+y^2}\right|.\end{aligned}$$

So we get

$$\begin{aligned}\ln\left|\frac{xy}{x^2+y^2}\right| = \ln|x| + C \Rightarrow \ln\left|\frac{y}{x^2+y^2}\right| = C \Rightarrow |y| = C'|x^2+y^2| \\ \Rightarrow x^2+y^2 = by,\end{aligned}$$

where $\mathbb{R}$. Hence the orthogonal trajectories of $x^2+y^2 = ax$ is $x^2+y^2 = by$.